2.6 Solving Inequalities Algebraically and Graphically

Properties of Inequalities

Simple inequalities were reviewed in Section P.1. There, the inequality symbols $<$, $\leq$, $>$, and $\geq$ were used to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality $x \geq 3$ denotes all real numbers x that are greater than or equal to 3.

In this section, you will study inequalities that contain more involved statements such as

$$5x - 7 > 3x + 9 \quad \text{and} \quad -3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. These values are **solutions** of the inequality and are said to **satisfy** the inequality. For instance, the number 9 is a solution of the first inequality listed above because

$$5(9) - 7 > 3(9) + 9$$
$$38 > 36.$$

On the other hand, the number 7 is not a solution because

$$5(7) - 7 \not> 3(7) + 9$$
$$28 \not> 30.$$

The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality.

The set of all points on the real number line that represent the solution set is the **graph of the inequality.** Graphs of many types of inequalities consist of intervals on the real number line.

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **properties of inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, *the direction of the inequality symbol must be reversed* in order to maintain a true statement. Here is an example.

$$-2 < 5 \qquad \text{Original inequality}$$
$$(-3)(-2) > (-3)(5) \qquad \text{Multiply each side by } -3 \text{ and reverse the inequality.}$$
$$6 > -15 \qquad \text{Simplify.}$$

Two inequalities that have the same solution set are **equivalent inequalities.** For instance, the inequalities

$$x + 2 < 5 \quad \text{and} \quad x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The properties listed at the top of the next page describe operations that can be used to create equivalent inequalities.

What you should learn

- Use properties of inequalities to solve linear inequalities.
- Solve inequalities involving absolute values.
- Solve polynomial inequalities.
- Solve rational inequalities.
- Use inequalities to model and solve real-life problems.

Why you should learn it

An inequality can be used to determine when a real-life quantity exceeds a given level. For instance, Exercises 85–88 on page 230 show how to use linear inequalities to determine when the number of hours per person spent playing video games exceeded the number of hours per person spent reading newspapers.

Image Source/Superstock

Prerequisite Skills

To review techniques for solving linear inequalities, see Appendix D.

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. *Transitive Property*

   $a < b$ and $b < c \Rightarrow a < c$

2. *Addition of Inequalities*

   $a < b$ and $c < d \Rightarrow a + c < b + d$

3. *Addition of a Constant*

   $a < b \Rightarrow a + c < b + c$

4. *Multiplying by a Constant*

   For $c > 0$, $a < b \Rightarrow ac < bc$

   For $c < 0$, $a < b \Rightarrow ac > bc$

Exploration

Use a graphing utility to graph $f(x) = 5x - 7$ and $g(x) = 3x + 9$ in the same viewing window. (Use $-1 \le x \le 15$ and $-5 \le y \le 50$.) For which values of x does the graph of f lie above the graph of g? Explain how the answer to this question can be used to solve the inequality in Example 1.

Each of the properties above is true if the symbol $<$ is replaced by $\le$ and $>$ is replaced by $\ge$. For instance, another form of Property 3 is as follows.

$$a \le b \Rightarrow a + c \le b + c$$

STUDY TIP

Checking the solution set of an inequality is not as simple as checking the solution(s) of an equation because there are simply too many x-values to substitute into the original inequality. However, you can get an indication of the validity of the solution set by substituting a few convenient values of x. For instance, in Example 1, try substituting $x = 5$ and $x = 10$ into the original inequality.

Solving a Linear Inequality

The simplest type of inequality to solve is a **linear inequality** in one variable, such as $2x + 3 > 4$. (See Appendix D for help with solving one-step linear inequalities.)

Example 1 Solving a Linear Inequality

Solve $5x - 7 > 3x + 9$.

Solution

$5x - 7 > 3x + 9$	Write original inequality.
$2x - 7 > 9$	Subtract $3x$ from each side.
$2x > 16$	Add 7 to each side.
$x > 8$	Divide each side by 2.

So, the solution set is all real numbers that are greater than 8. The interval notation for this solution set is $(8, \infty)$. The number line graph of this solution set is shown in Figure 2.45. Note that a parenthesis at 8 on the number line indicates that 8 *is not* part of the solution set.

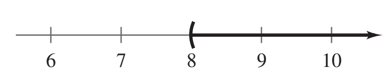


Figure 2.45 *Solution Interval:* $(8, \infty)$

✓CHECKPOINT Now try Exercise 13.

Note that the four inequalities forming the solution steps of Example 1 are all *equivalent* in the sense that each has the same solution set.

Example 2 Solving an Inequality

Solve $1 - \frac{3}{2}x \geq x - 4$.

Algebraic Solution

$1 - \frac{3}{2}x \geq x - 4$ Write original inequality.

$2 - 3x \geq 2x - 8$ Multiply each side by the LCD.

$2 - 5x \geq -8$ Subtract $2x$ from each side.

$-5x \geq -10$ Subtract 2 from each side.

$x \leq 2$ Divide each side by -5 and reverse the inequality.

The solution set is all real numbers that are less than or equal to 2. The interval notation for this solution set is $(-\infty, 2]$. The number line graph of this solution set is shown in Figure 2.46. Note that a bracket at 2 on the number line indicates that 2 *is* part of the solution set.

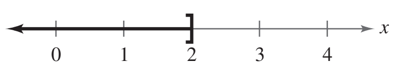


Figure 2.46 ***Solution Interval:*** $(-\infty, 2]$

CHECKPOINT Now try Exercise 15.

Graphical Solution

Use a graphing utility to graph $y_1 = 1 - \frac{3}{2}x$ and $y_2 = x - 4$ in the same viewing window. In Figure 2.47, you can see that the graphs appear to intersect at the point $(2, -2)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies above the graph of y_2 to the left of their point of intersection, which implies that $y_1 \geq y_2$ for all $x \leq 2$.

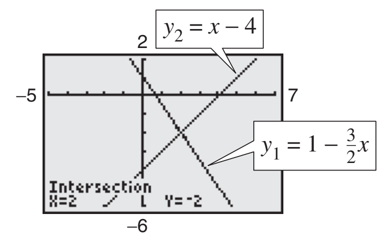


Figure 2.47

Sometimes it is possible to write two inequalities as a **double inequality,** as demonstrated in Example 3.

Example 3 Solving a Double Inequality

Solve $-3 \leq 6x - 1$ and $6x - 1 < 3$.

Algebraic Solution

$-3 \leq 6x - 1 < 3$ Write as a double inequality.

$-2 \leq 6x < 4$ Add 1 to each part.

$-\frac{1}{3} \leq x < \frac{2}{3}$ Divide by 6 and simplify.

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ *and* less than $\frac{2}{3}$. The interval notation for this solution set is $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The number line graph of this solution set is shown in Figure 2.48.

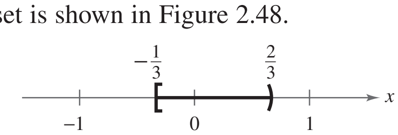


Figure 2.48 ***Solution Interval:*** $\left[-\frac{1}{3}, \frac{2}{3}\right)$

CHECKPOINT Now try Exercise 17.

Graphical Solution

Use a graphing utility to graph $y_1 = 6x - 1$, $y_2 = -3$, and $y_3 = 3$ in the same viewing window. In Figure 2.49, you can see that the graphs appear to intersect at the points $\left(-\frac{1}{3}, -3\right)$ and $\left(\frac{2}{3}, 3\right)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies above the graph of y_2 to the right of $\left(-\frac{1}{3}, -3\right)$ *and* the graph of y_1 lies below the graph of y_3 to the left of $\left(\frac{2}{3}, 3\right)$. This implies that $y_2 \leq y_1 < y_3$ when $-\frac{1}{3} \leq x < \frac{2}{3}$.

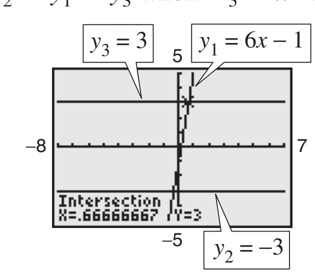


Figure 2.49

Inequalities Involving Absolute Values

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a.

   $|x| < a$ if and only if $-a < x < a$. Double inequality

2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a.

   $|x| > a$ if and only if $x < -a$ or $x > a$. Compound inequality

These rules are also valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example 4 Solving Absolute Value Inequalities

You may need to remind your students that $|x| = a$ corresponds to $x = a$ *or* $x = -a$.

Solve each inequality.

a. $|x - 5| < 2$ **b.** $|x - 5| > 2$

Algebraic Solution

a. $|x - 5| < 2$ Write original inequality.

$-2 < x - 5 < 2$ Write double inequality.

$3 < x < 7$ Add 5 to each part.

The solution set is all real numbers that are greater than 3 *and* less than 7. The interval notation for this solution set is $(3, 7)$. The number line graph of this solution set is shown in Figure 2.50.

b. The absolute value inequality $|x - 5| > 2$ is equivalent to the following compound inequality: $x - 5 < -2$ *or* $x - 5 > 2$.

Solve first inequality: $x - 5 < -2$ Write first inequality.

$x < 3$ Add 5 to each side.

Solve second inequality: $x - 5 > 2$ Write second inequality.

$x > 7$ Add 5 to each side.

The solution set is all real numbers that are less than 3 *or* greater than 7. The interval notation for this solution set is $(-\infty, 3) \cup (7, \infty)$. The symbol $\cup$ is called a *union* symbol and is used to denote the combining of two sets. The number line graph of this solution set is shown in Figure 2.51.

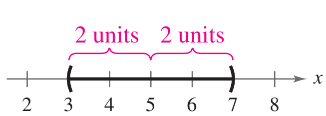


Figure 2.50

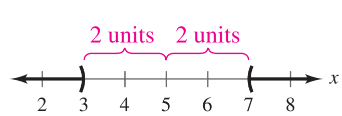


Figure 2.51

CHECKPOINT Now try Exercise 31.

Graphical Solution

a. Use a graphing utility to graph $y_1 = |x - 5|$ and $y_2 = 2$ in the same viewing window. In Figure 2.52, you can see that the graphs appear to intersect at the points $(3, 2)$ and $(7, 2)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies below the graph of y_2 when $3 < x < 7$. So, you can approximate the solution set to be all real numbers greater than 3 *and* less than 7.

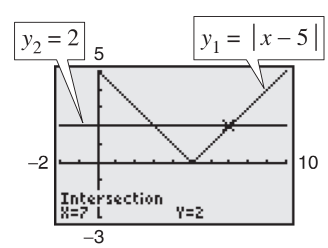


Figure 2.52

b. In Figure 2.52, you can see that the graph of y_1 lies above the graph of y_2 when $x < 3$ *or* when $x > 7$. So, you can approximate the solution set to be all real numbers that are less than 3 *or* greater than 7.

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, use the fact that a polynomial can change signs only at its zeros (the x-values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **critical numbers** of the inequality, and the resulting open intervals are the **test intervals** for the inequality. For instance, the polynomial above factors as

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

and has two zeros, $x = -1$ and $x = 3$, which divide the real number line into three test intervals: $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$. To solve the inequality $x^2 - 2x - 3 < 0$, you need to test only one value in each test interval.

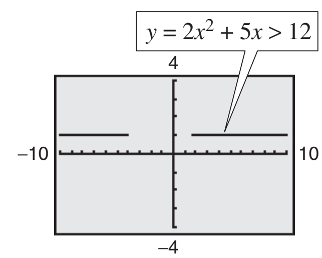
Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order. The zeros of a polynomial are its critical numbers.
2. Use the critical numbers to determine the test intervals.
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, the polynomial will have negative values for *every* x-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for *every* x-value in the interval.

TECHNOLOGY TIP

Some graphing utilities will produce graphs of inequalities. For instance, you can graph $2x^2 + 5x > 12$ by setting the graphing utility to *dot* mode and entering $y = 2x^2 + 5x > 12$. Using the settings $-10 \le x \le 10$ and $-4 \le y \le 4$, your graph should look like the graph shown below. Solve the problem algebraically to verify that the solution is $(-\infty, -4) \cup \left(\frac{3}{2}, \infty\right)$.



Example 5 Investigating Polynomial Behavior

To determine the intervals on which $x^2 - 3$ is entirely negative and those on which it is entirely positive, factor the quadratic as $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$. The critical numbers occur at $x = -\sqrt{3}$ and $x = \sqrt{3}$. So, the test intervals for the quadratic are $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$, and $(\sqrt{3}, \infty)$. In each test interval, choose a representative x-value and evaluate the polynomial, as shown in the table.

Interval	x-Value	Value of Polynomial	Sign of Polynomial
$(-\infty, -\sqrt{3})$	$x = -3$	$(-3)^2 - 3 = 6$	Positive
$(-\sqrt{3}, \sqrt{3})$	$x = 0$	$(0)^2 - 3 = -3$	Negative
$(\sqrt{3}, \infty)$	$x = 5$	$(5)^2 - 3 = 22$	Positive

The polynomial has negative values for every x in the interval $(-\sqrt{3}, \sqrt{3})$ and positive values for every x in the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. This result is shown graphically in Figure 2.53.

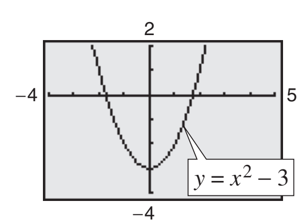


Figure 2.53

CHECKPOINT Now try Exercise 49.

To determine the test intervals for a polynomial inequality, the inequality must first be written in general form with the polynomial on one side.

Example 6 Solving a Polynomial Inequality

Solve $2x^2 + 5x > 12$.

Algebraic Solution

$2x^2 + 5x - 12 > 0$ Write inequality in general form.

$(x + 4)(2x - 3) > 0$ Factor.

Critical Numbers: $x = -4,\ x = \frac{3}{2}$

Test Intervals: $(-\infty, -4),\ \left(-4, \frac{3}{2}\right),\ \left(\frac{3}{2}, \infty\right)$

Test: Is $(x + 4)(2x - 3) > 0$?

After testing these intervals, you can see that the polynomial $2x^2 + 5x - 12$ is positive on the open intervals $(-\infty, -4)$ and $\left(\frac{3}{2}, \infty\right)$. Therefore, the solution set of the inequality is

$$(-\infty, -4) \cup \left(\tfrac{3}{2}, \infty\right).$$

CHECKPOINT Now try Exercise 55.

Graphical Solution

First write the polynomial inequality $2x^2 + 5x > 12$ as $2x^2 + 5x - 12 > 0$. Then use a graphing utility to graph $y = 2x^2 + 5x - 12$. In Figure 2.54, you can see that the graph is *above* the x-axis when x is less than -4 *or* when x is greater than $\frac{3}{2}$. So, you can graphically approximate the solution set to be $(-\infty, -4) \cup \left(\frac{3}{2}, \infty\right)$.

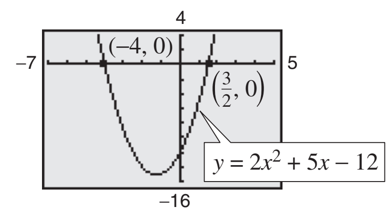


Figure 2.54

Example 7 Solving a Polynomial Inequality

Solve $2x^3 - 3x^2 - 32x > -48$.

Solution

$2x^3 - 3x^2 - 32x + 48 > 0$ Write inequality in general form.

$x^2(2x - 3) - 16(2x - 3) > 0$ Factor by grouping.

$(x^2 - 16)(2x - 3) > 0$ Distributive Property

$(x - 4)(x + 4)(2x - 3) > 0$ Factor difference of two squares.

The critical numbers are $x = -4$, $x = \frac{3}{2}$, and $x = 4$; and the test intervals are $(-\infty, -4)$, $\left(-4, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 4\right)$, and $(4, \infty)$.

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, -4)$	$x = -5$	$2(-5)^3 - 3(-5)^2 - 32(-5) + 48 = -117$	Negative
$\left(-4, \frac{3}{2}\right)$	$x = 0$	$2(0)^3 - 3(0)^2 - 32(0) + 48 = 48$	Positive
$\left(\frac{3}{2}, 4\right)$	$x = 2$	$2(2)^3 - 3(2)^2 - 32(2) + 48 = -12$	Negative
$(4, \infty)$	$x = 5$	$2(5)^3 - 3(5)^2 - 32(5) + 48 = 63$	Positive

From this you can conclude that the polynomial is positive on the open intervals $\left(-4, \frac{3}{2}\right)$ and $(4, \infty)$. So, the solution set is $\left(-4, \frac{3}{2}\right) \cup (4, \infty)$.

CHECKPOINT Now try Exercise 61.

STUDY TIP

When solving a quadratic inequality, be sure you have accounted for the particular type of inequality symbol given in the inequality. For instance, in Example 7, note that the original inequality contained a "greater than" symbol and the solution consisted of two open intervals. If the original inequality had been

$$2x^3 - 3x^2 - 32x \ge -48$$

the solution would have consisted of the closed interval $\left[-4, \frac{3}{2}\right]$ and the interval $[4, \infty)$.

Example 8 Unusual Solution Sets

a. The solution set of

$$x^2 + 2x + 4 > 0$$

consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the value of the quadratic $x^2 + 2x + 4$ is positive for every real value of x, as indicated in Figure 2.55(a). (Note that this quadratic inequality has *no* critical numbers. In such a case, there is only one test interval—the entire real number line.)

b. The solution set of

$$x^2 + 2x + 1 \le 0$$

consists of the single real number $\{-1\}$, because the quadratic $x^2 + 2x + 1$ has one critical number, $x = -1$, and it is the only value that satisfies the inequality, as indicated in Figure 2.55(b).

c. The solution set of

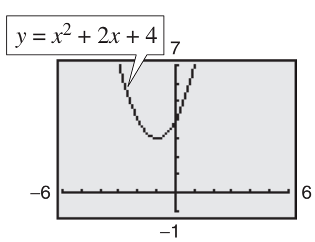
$$x^2 + 3x + 5 < 0$$

is empty. In other words, the quadratic $x^2 + 3x + 5$ is not less than zero for any value of x, as indicated in Figure 2.55(c).

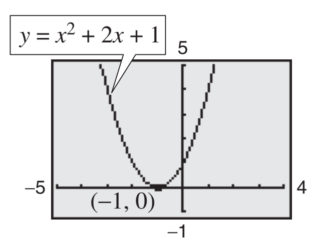
d. The solution set of

$$x^2 - 4x + 4 > 0$$

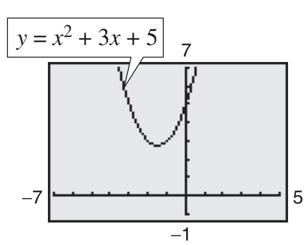
consists of all real numbers *except* the number 2. In interval notation, this solution set can be written as $(-\infty, 2) \cup (2, \infty)$. The graph of $x^2 - 4x + 4$ lies above the x-axis except at $x = 2$, where it touches it, as indicated in Figure 2.55(d).



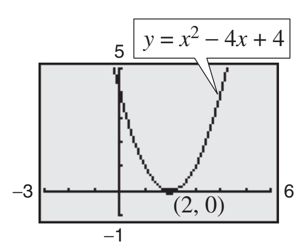
(a)



(b)



(c)



(d)

Figure 2.55

CHECKPOINT Now try Exercise 59.

TECHNOLOGY TIP

One of the advantages of technology is that you can solve complicated polynomial inequalities that might be difficult, or even impossible, to factor. For instance, you could use a graphing utility to approximate the solution to the inequality

$$x^3 - 0.2x^2 - 3.16x + 1.4 < 0.$$

Remind students that they can check the answers to inequality problems in two ways:

Algebraically: Substitute x-values into the original inequality.

Graphically: Sketch the graph of the polynomial written in standard form and note where the graph lies relative to the x-axis.

Students can also use a graphing utility to check their answers, as indicated in the Technology Tip on page 223.

Rational Inequalities

The concepts of critical numbers and test intervals can be extended to inequalities involving rational expressions. To do this, use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the *critical numbers* of a rational inequality.

Example 9 Solving a Rational Inequality

Solve $\dfrac{2x - 7}{x - 5} \le 3.$

Algebraic Solution

$$\frac{2x - 7}{x - 5} \le 3 \qquad \text{Write original inequality.}$$

$$\frac{2x - 7}{x - 5} - 3 \le 0 \qquad \text{Write in general form.}$$

$$\frac{2x - 7 - 3x + 15}{x - 5} \le 0 \qquad \text{Write as single fraction.}$$

$$\frac{-x + 8}{x - 5} \le 0 \qquad \text{Simplify.}$$

Now, in standard form you can see that the critical numbers are $x = 5$ and $x = 8$, and you can proceed as follows.

Critical Numbers: $x = 5, x = 8$
Test Intervals: $(-\infty, 5), (5, 8), (8, \infty)$
Test: Is $\dfrac{-x + 8}{x - 5} \le 0$?

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, 5)$	$x = 0$	$\dfrac{-0 + 8}{0 - 5} = -\dfrac{8}{5}$	Negative
$(5, 8)$	$x = 6$	$\dfrac{-6 + 8}{6 - 5} = 2$	Positive
$(8, \infty)$	$x = 9$	$\dfrac{-9 + 8}{9 - 5} = -\dfrac{1}{4}$	Negative

By testing these intervals, you can determine that the rational expression $(-x + 8)/(x - 5)$ is negative in the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because $(-x + 8)/(x - 5) = 0$ when $x = 8$, you can conclude that the solution set of the inequality is $(-\infty, 5) \cup [8, \infty)$.

CHECKPOINT Now try Exercise 69.

Graphical Solution

Use a graphing utility to graph

$$y_1 = \frac{2x - 7}{x - 5} \quad \text{and} \quad y_2 = 3$$

in the same viewing window. In Figure 2.56, you can see that the graphs appear to intersect at the point $(8, 3)$. Use the *intersect* feature of the graphing utility to confirm this. The graph of y_1 lies below the graph of y_2 in the intervals $(-\infty, 5)$ and $[8, \infty)$. So, you can graphically approximate the solution set to be all real numbers less than 5 *or* greater than or equal to 8.

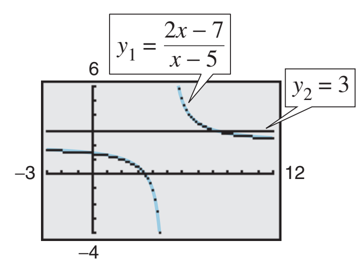


Figure 2.56

Note in Example 9 that $x = 5$ is not included in the solution set because the inequality is undefined when $x = 5$.

Application

In Section 1.3 you studied the *implied domain* of a function, the set of all x-values for which the function is defined. A common type of implied domain is used to avoid even roots of negative numbers, as shown in Example 10.

Example 10 Finding the Domain of an Expression

Find the domain of $\sqrt{64 - 4x^2}$.

Solution

Because $\sqrt{64 - 4x^2}$ is defined only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \ge 0$.

$64 - 4x^2 \ge 0$ — Write in general form.

$16 - x^2 \ge 0$ — Divide each side by 4.

$(4 - x)(4 + x) \ge 0$ — Factor.

The inequality has two critical numbers: $x = -4$ and $x = 4$. A test shows that $64 - 4x^2 \ge 0$ in the *closed interval* $[-4, 4]$. The graph of $y = \sqrt{64 - 4x^2}$, shown in Figure 2.57, confirms that the domain is $[-4, 4]$.

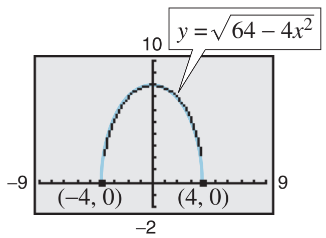


Figure 2.57

CHECKPOINT Now try Exercise 77.

Example 11 Height of a Projectile

A projectile is fired straight upward from ground level with an initial velocity of 384 feet per second. During what time period will its height exceed 2000 feet?

Solution

In Section 2.4 you saw that the position of an object moving vertically can be modeled by the *position equation*

$$s = -16t^2 + v_0t + s_0$$

where s is the height in feet and t is the time in seconds. In this case, $s_0 = 0$ and $v_0 = 384$. So, you need to solve the inequality $-16t^2 + 384t > 2000$. Using a graphing utility, graph $y_1 = -16t^2 + 384t$ and $y_2 = 2000$, as shown in Figure 2.58. From the graph, you can determine that $-16t^2 + 384t > 2000$ for t between approximately 7.6 and 16.4. You can verify this result algebraically.

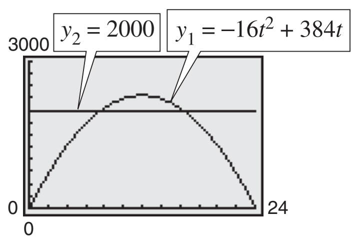


Figure 2.58

$-16t^2 + 384t > 2000$ — Write original inequality.

$t^2 - 24t < -125$ — Divide by -16 and reverse inequality.

$t^2 - 24t + 125 < 0$ — Write in general form.

By the Quadratic Formula the critical numbers are $t = 12 - \sqrt{19}$ and $t = 12 + \sqrt{19}$, or approximately 7.64 and 16.36. A test will verify that the height of the projectile will exceed 2000 feet when $7.64 < t < 16.36$; that is, during the time interval (7.64, 16.36) seconds.

CHECKPOINT Now try Exercise 81.

2.6 Exercises

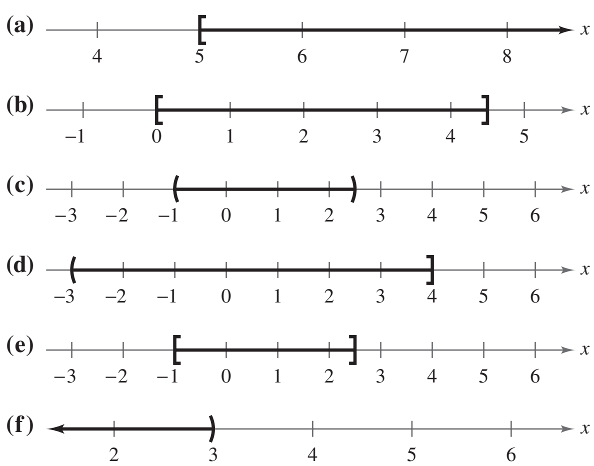
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. To solve a linear inequality in one variable, you can use the properties of inequalities, which are identical to those used to solve an equation, with the exception of multiplying or dividing each side by a _______ constant.
2. It is sometimes possible to write two inequalities as one inequality, called a _______ inequality.
3. The solutions to $|x| \le a$ are those values of x such that _______ .
4. The solutions to $|x| \ge a$ are those values of x such that _______ or _______ .
5. The critical numbers of a rational expression are its _______ and its _______ .

In Exercises 1–6, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



1. $x < 3$
2. $x \ge 5$
3. $-3 < x \le 4$
4. $0 \le x \le \frac{9}{2}$
5. $-1 \le x \le \frac{5}{2}$
6. $-1 < x < \frac{5}{2}$

In Exercises 7–10, determine whether each value of x is a solution of the inequality.

Inequality	*Values*	
7. $5x - 12 > 0$	(a) $x = 3$	(b) $x = -3$
	(c) $x = \frac{5}{2}$	(d) $x = \frac{3}{2}$
8. $-5 < 2x - 1 \le 1$	(a) $x = -\frac{1}{2}$	(b) $x = -\frac{5}{2}$
	(c) $x = \frac{4}{3}$	(d) $x = 0$
9. $-1 < \frac{3 - x}{2} \le 1$	(a) $x = 0$	(b) $x = \sqrt{5}$
	(c) $x = 1$	(d) $x = 5$
10. $\lvert x - 10 \rvert \ge 3$	(a) $x = 13$	(b) $x = -1$
	(c) $x = 14$	(d) $x = 9$

In Exercises 11–20, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.

11. $-10x < 40$
12. $6x > 15$
13. $4(x + 1) < 2x + 3$
14. $2x + 7 < 3(x - 4)$
15. $\frac{3}{4}x - 6 \le x - 7$
16. $3 + \frac{2}{7}x > x - 2$
17. $-8 \le 1 - 3(x - 2) < 13$
18. $0 \le 2 - 3(x + 1) < 20$
19. $-4 < \frac{2x - 3}{3} < 4$
20. $0 \le \frac{x + 3}{2} < 5$

Graphical Analysis **In Exercises 21–24, use a graphing utility to approximate the solution.**

21. $5 - 2x \ge 1$
22. $20 < 6x - 1$
23. $3(x + 1) < x + 7$
24. $4(x - 3) \le 8 - x$

In Exercises 25–28, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

Equation	*Inequalities*	
25. $y = 2x - 3$	(a) $y \ge 1$	(b) $y \le 0$
26. $y = -3x + 8$	(a) $-1 \le y \le 3$	(b) $y \le 0$
27. $y = -\frac{1}{2}x + 2$	(a) $0 \le y \le 3$	(b) $y \ge 0$
28. $y = \frac{2}{3}x + 1$	(a) $y \le 5$	(b) $y \ge 0$

In Exercises 29–36, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solutions graphically.

29. $|5x| > 10$ **30.** $\left|\frac{x}{2}\right| \le 1$

31. $|x - 7| < 6$ **32.** $|x - 20| \ge 4$

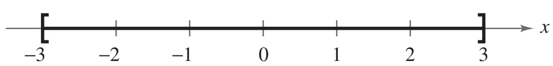
33. $|x + 14| + 3 > 17$ **34.** $\left|\frac{x - 3}{2}\right| \ge 5$

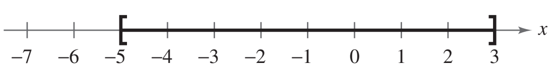
35. $10|1 - 2x| < 5$ **36.** $3|4 - 5x| \le 9$

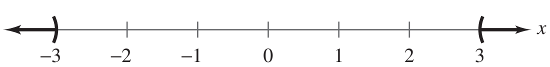
In Exercises 37 and 38, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

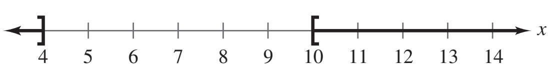
	Equation	*Inequalities*	
37.	$y = \lvert x - 3 \rvert$	(a) $y \le 2$	(b) $y \ge 4$
38.	$y = \lvert \frac{1}{2}x + 1 \rvert$	(a) $y \le 4$	(b) $y \ge 1$

In Exercises 39–46, use absolute value notation to define the interval (or pair of intervals) on the real number line.

39.


40.


41.


42.


43. All real numbers within 10 units of 7

44. All real numbers no more than 8 units from −5

45. All real numbers at least 5 units from 3

46. All real numbers more than 3 units from −1

In Exercises 47–52, determine the intervals on which the polynomial is entirely negative and those on which it is entirely positive.

47. $x^2 - 4x - 5$ **48.** $x^2 - 3x - 4$

49. $2x^2 - 4x - 3$ **50.** $2x^2 - x - 5$

51. $x^2 - 4x + 5$ **52.** $-x^2 + 6x - 10$

In Exercises 53–62, solve the inequality and graph the solution on the real number line. Use a graphing utility to verify your solution graphically.

53. $(x + 2)^2 < 25$ **54.** $(x - 3)^2 \ge 1$

55. $x^2 + 4x + 4 \ge 9$ **56.** $x^2 - 6x + 9 < 16$

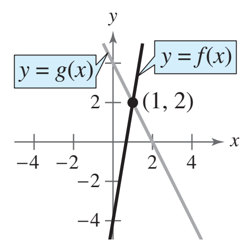
57. $x^3 - 4x \ge 0$ **58.** $x^4(x - 3) \le 0$

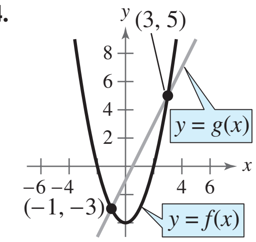
59. $3x^2 - 11x + 16 \le 0$ **60.** $4x^2 + 12x + 9 \le 0$

61. $2x^3 + 5x^2 > 6x + 9$ **62.** $2x^3 + 3x^2 < 11x + 6$

In Exercises 63 and 64, use the graph of the function to solve the equation or inequality.

(a) $f(x) = g(x)$ **(b)** $f(x) \ge g(x)$ **(c)** $f(x) > g(x)$

63.


64.


In Exercises 65 and 66, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

	Equation	*Inequalities*	
65.	$y = -x^2 + 2x + 3$	(a) $y \le 0$	(b) $y \ge 3$
66.	$y = x^3 - x^2 - 16x + 16$	(a) $y \le 0$	(b) $y \ge 36$

In Exercises 67–70, solve the inequality and graph the solution on the real number line. Use a graphing utility to verify your solution graphically.

67. $\frac{1}{x} - x > 0$ **68.** $\frac{1}{x} - 4 < 0$

69. $\frac{x + 6}{x + 1} - 2 < 0$ **70.** $\frac{x + 12}{x + 2} - 3 \ge 0$

In Exercises 71 and 72, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

	Equation	*Inequalities*	
71.	$y = \frac{3x}{x - 2}$	(a) $y \le 0$	(b) $y \ge 6$
72.	$y = \frac{5x}{x^2 + 4}$	(a) $y \ge 1$	(b) $y \le 0$

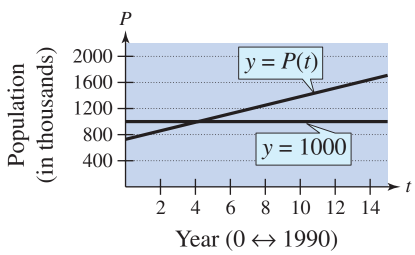
In Exercises 73–78, find the domain of x in the expression.

73. $\sqrt{x - 5}$ **74.** $\sqrt[4]{6x + 15}$

75. $\sqrt[3]{6 - x}$ **76.** $\sqrt[3]{2x^2 - 8}$

77. $\sqrt{x^2 - 4}$ **78.** $\sqrt[4]{4 - x^2}$

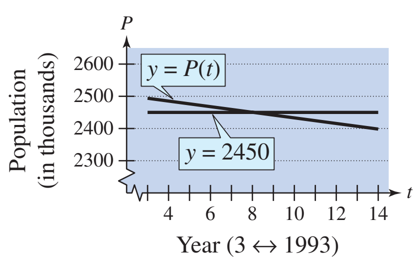
79. ***Population*** The graph models the population P (in thousands) of Las Vegas, Nevada from 1990 to 2004, where t is the year, with $t = 0$ corresponding to 1990. Also shown is the line $y = 1000$. Use the graphs of the model and the horizontal line to write an equation or an inequality that could be solved to answer the question. Then answer the question. (Source: U.S. Census Bureau)



(a) In what year does the population of Las Vegas reach one million?

(b) Over what time period is the population of Las Vegas less than one million? greater than one million?

80. ***Population*** The graph models the population P (in thousands) of Pittsburgh, Pennsylvania from 1993 to 2004, where t is the year, with $t = 3$ corresponding to 1993. Also shown is the line $y = 2450$. Use the graphs of the model and the horizontal line to write an equation or an inequality that could be solved to answer the question. Then answer the question. (Source: U.S. Census Bureau)



(a) In what year did the population of Pittsburgh equal 2.45 million?

(b) Over what time period is the population of Pittsburgh less than 2.45 million? greater than 2.45 million?

81. ***Height of a Projectile*** A projectile is fired straight upward from ground level with an initial velocity of 160 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height exceed 384 feet?

82. ***Height of a Projectile*** A projectile is fired straight upward from ground level with an initial velocity of 128 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height be less than 128 feet?

83. ***Education*** The numbers D of doctorate degrees (in thousands) awarded to female students from 1990 to 2003 in the United States can be approximated by the model $D = -0.0165t^2 + 0.755t + 14.06,\ 0 \le t \le 13$, where t is the year, with $t = 0$ corresponding to 1990. (Source: U.S. National Center for Education Statistics)

(a) Use a graphing utility to graph the model.

(b) Use the *zoom* and *trace* features to find when the number of degrees was between 15 and 20 thousand.

(c) Algebraically verify your results from part (b).

(d) According to the model, will the number of degrees exceed 30 thousand? If so, when? If not, explain.

84. ***Data Analysis*** You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). Sample data from 12 athletes is shown below.

(165, 170), (184, 185), (150, 200), (210, 255), (196, 205), (240, 295), (202, 190), (170, 175), (185, 195), (190, 185), (230, 250), (160, 150)

(a) Use a graphing utility to plot the data.

(b) A model for this data is $y = 1.3x - 36$. Use a graphing utility to graph the equation in the same viewing window used in part (a).

(c) Use the graph to estimate the value of x that predict a maximum bench-press weight of at least 200 pounds.

(d) Use the graph to write a statement about the accuracy of the model. If you think the graph indicates that an athlete's weight is not a good indicator of the athlete's maximum bench-press weight, list other factors that might influence an individual's maximum bench-press weight.

Leisure Time **In Exercises 85–88, use the models below which approximate the annual numbers of hours per person spent reading daily newspapers N and playing video games V for the years 2000 to 2005, where t is the year, with $t = 0$ corresponding to 2000.** (Source: Veronis Suhler Stevenson)

Daily Newspapers: $N = -2.51t + 179.6, \quad 0 \le t \le 5$

Video Games: $V = 3.37t + 57.9, \quad 0 \le t \le 5$

85. Solve the inequality $V(t) \ge 65$. Explain what the solution of the inequality represents.

86. Solve the inequality $N(t) \le 175$. Explain what the solution of the inequality represents.

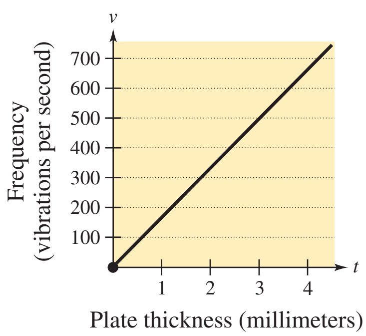
87. Solve the equation $V(t) = N(t)$. Explain what the solution of the equation represents.

88. Solve the inequality $V(t) > N(t)$. Explain what the solution of the inequality represents.

Music **In Exercises 89–92, use the following information. Michael Kasha of Florida State University used physics and mathematics to design a new classical guitar. He used the model for the frequency of the vibrations on a circular plate**

$$v = \frac{2.6t}{d^2}\sqrt{\frac{E}{\rho}}$$

where v is the frequency (in vibrations per second), t is the plate thickness (in millimeters), d is the diameter of the plate, E is the elasticity of the plate material, and ρ is the density of the plate material. For fixed values of d, E, and ρ, the graph of the equation is a line, as shown in the figure.



89. Estimate the frequency when the plate thickness is 2 millimeters.

90. Estimate the plate thickness when the frequency is 600 vibrations per second.

91. Approximate the interval for the plate thickness when the frequency is between 200 and 400 vibrations per second.

92. Approximate the interval for the frequency when the plate thickness is less than 3 millimeters.

In Exercises 93 and 94, (a) write equations that represent each option, (b) use a graphing utility to graph the options in the same viewing window, (c) determine when each option is the better choice, and (d) explain which option you would choose.

93. ***Cellular Phones*** You are trying to decide between two different cellular telephone contracts, option A and option B. Option A has a monthly fee of \$12 plus \$0.15 per minute. Option B has no monthly fee but charges \$0.20 per minute. All other monthly charges are identical.

94. ***Moving*** You are moving from your home to your dorm room, and the moving company has offered you two options. The charges for gasoline, insurance, and all other incidental fees are equal.

Option A: \$200 plus \$18 per hour to move all of your belongings from your home to your dorm room.

Option B: \$24 per hour to move all of your belongings from your home to your dorm room.

Synthesis

True or False? **In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.**

95. If $-10 \le x \le 8$, then $-10 \ge -x$ and $-x \ge -8$.

96. The solution set of the inequality $\frac{3}{2}x^2 + 3x + 6 \ge 0$ is the entire set of real numbers.

In Exercises 97 and 98, consider the polynomial $(x - a)(x - b)$ and the real number line (see figure).

97. Identify the points on the line where the polynomial is zero.

98. In each of the three subintervals of the line, write the sign of each factor and the sign of the product. For which x-values does the polynomial possibly change signs?

99. ***Proof*** The arithmetic mean of a and b is given by $(a + b)/2$. Order the statements of the proof to show that if $a < b$, then $a < (a + b)/2 < b$.

i. $a < \frac{a + b}{2} < b$

ii. $2a < 2b$

iii. $2a < a + b < 2b$

iv. $a < b$

100. ***Proof*** The geometric mean of a and b is given by $\sqrt{ab}$. Order the statements of the proof to show that if $0 < a < b$, then $a < \sqrt{ab} < b$.

i. $a^2 < ab < b^2$

ii. $0 < a < b$

iii. $a < \sqrt{ab} < b$

Skills Review

In Exercises 101–104, sketch a graph of the function.

101. $f(x) = -x^2 + 6$

102. $f(x) = \frac{1}{3}(x - 5)^2$

103. $f(x) = -|x + 5| - 6$

104. $f(x) = \frac{1}{2}|-x| - 4$

In Exercises 105–108, find the inverse function.

105. $y = 12x$

106. $y = 5x + 8$

107. $y = x^3 + 7$

108. $y = \sqrt[3]{x - 7}$

109. ***Make a Decision*** To work an extended application analyzing the number of heart disease deaths per 100,000 people in the United States, visit this textbook's *Online Study Center*. (Data Source: U.S. National Center for Health Statistics)