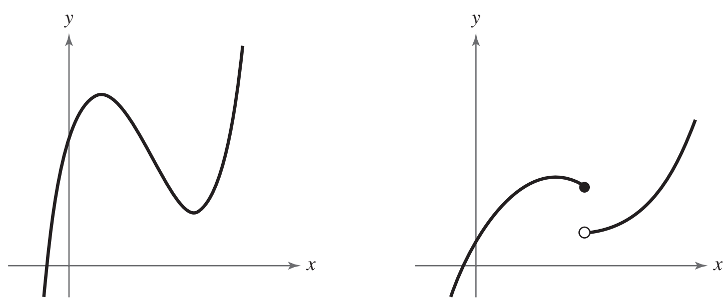
3.2 Polynomial Functions of Higher Degree

Graphs of Polynomial Functions

You should be able to sketch accurate graphs of polynomial functions of degrees 0, 1, and 2. The graphs of polynomial functions of degree greater than 2 are more difficult to sketch by hand. However, in this section you will learn how to recognize some of the basic features of the graphs of polynomial functions. Using these features along with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

The graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 3.14. Informally, you can say that a function is continuous if its graph can be drawn with a pencil without lifting the pencil from the paper.

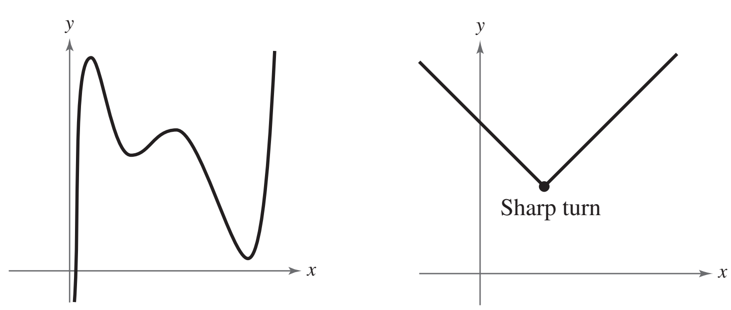


(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 3.14

Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 3.15(a). It cannot have a sharp turn such as the one shown in Figure 3.15(b).



(a) Polynomial functions have graphs with smooth, rounded turns.

(b) Functions with graphs that have sharp turns are not polynomial functions.

Figure 3.15

What you should learn

- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions.
- Find and use zeros of polynomial functions as sketching aids.
- Use the Intermediate Value Theorem to help locate zeros of polynomial functions.

Why you should learn it

You can use polynomial functions to model various aspects of nature, such as the growth of a red oak tree, as shown in Exercise 94 on page 274.

Leonard Lee Rue III/Earth Scenes

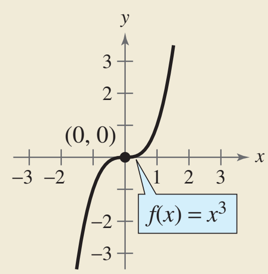
Library of Parent Functions: Polynomial Function

The graphs of polynomial functions of degree 1 are lines, and those of functions of degree 2 are parabolas. The graphs of all polynomial functions are smooth and continuous. A polynomial function of degree n has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

where n is a positive integer and $a_n \neq 0$. The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero. If n is even, the graph is similar to the graph of $f(x) = x^2$ and touches the axis at the x-intercept. If n is odd, the graph is similar to the graph of $f(x) = x^3$ and crosses the axis at the x-intercept. The greater the value of n, the flatter the graph near the origin. The basic characteristics of the *cubic function* $f(x) = x^3$ are summarized below. A review of polynomial functions can be found in the *Study Capsules*.

Graph of $f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing on $(-\infty, \infty)$
Odd function
Origin symmetry



Exploration

Use a graphing utility to graph $y = x^n$ for $n = 2, 4,$ and 8. (Use the viewing window $-1.5 \leq x \leq 1.5$ and $-1 \leq y \leq 6$.) Compare the graphs. In the interval $(-1, 1)$, which graph is on the bottom? Outside the interval $(-1, 1)$, which graph is on the bottom?

Use a graphing utility to graph $y = x^n$ for $n = 3, 5,$ and 7. (Use the viewing window $-1.5 \leq x \leq 1.5$ and $-4 \leq y \leq 4$.) Compare the graphs. In the intervals $(-\infty, -1)$ and $(0, 1)$, which graph is on the bottom? In the intervals $(-1, 0)$ and $(1, \infty)$, which graph is on the bottom?

Example 1 Transformations of Monomial Functions

Sketch the graphs of (a) $f(x) = -x^5$, (b) $g(x) = x^4 + 1$, and (c) $h(x) = (x + 1)^4$.

Solution

a. Because the degree of $f(x) = -x^5$ is odd, the graph is similar to the graph of $y = x^3$. Moreover, the negative coefficient reflects the graph in the x-axis, as shown in Figure 3.16.

b. The graph of $g(x) = x^4 + 1$ is an upward shift of one unit of the graph of $y = x^4$, as shown in Figure 3.17.

c. The graph of $h(x) = (x + 1)^4$ is a left shift of one unit of the graph of $y = x^4$, as shown in Figure 3.18.

Prerequisite Skills

If you have difficulty with this example, review shifting and reflecting of graphs in Section 1.5.

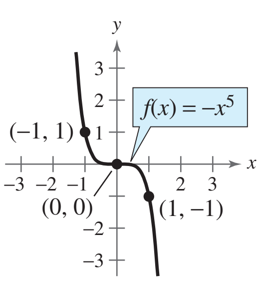


Figure 3.16

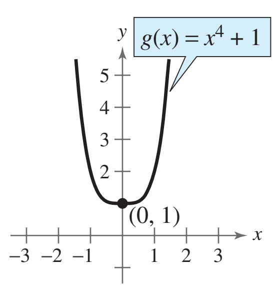


Figure 3.17

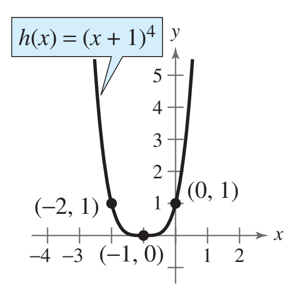


Figure 3.18

CHECKPOINT Now try Exercise 9.

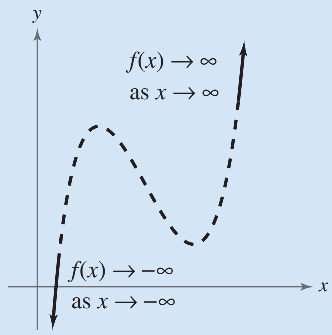
The Leading Coefficient Test

In Example 1, note that all three graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial eventually rises or falls can be determined by the polynomial function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

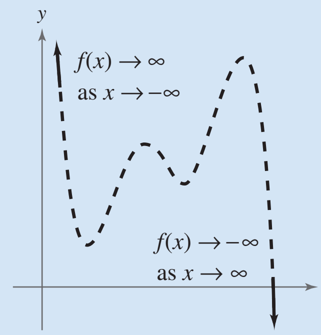
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0$, $a_n \neq 0$, eventually rises or falls in the following manner.

1. When n is odd:

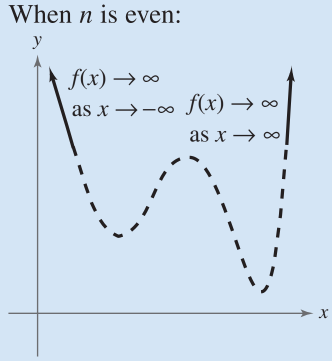


If the leading coefficient is positive ($a_n > 0$), the graph falls to the left and rises to the right.

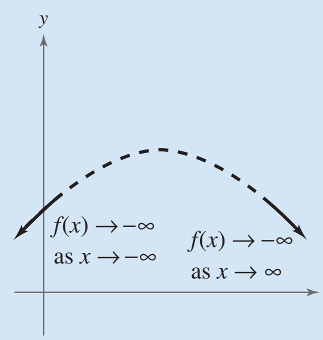


If the leading coefficient is negative ($a_n < 0$), the graph rises to the left and falls to the right.

2. When n is even:



If the leading coefficient is positive ($a_n > 0$), the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), the graph falls to the left and right.

Note that the dashed portions of the graphs indicate that the test determines only the right-hand and left-hand behavior of the graph.

As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

Exploration

For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree and sign of the leading coefficient of the function and the right- and left-hand behavior of the graph of the function.

a. $y = x^3 - 2x^2 - x + 1$
b. $y = 2x^5 + 2x^2 - 5x + 1$
c. $y = -2x^5 - x^2 + 5x + 3$
d. $y = -x^3 + 5x - 2$
e. $y = 2x^2 + 3x - 4$
f. $y = x^4 - 3x^2 + 2x - 1$
g. $y = -x^2 + 3x + 2$
h. $y = -x^6 - x^2 - 5x + 4$

STUDY TIP

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

A review of the shapes of the graphs of polynomial functions of degrees 0, 1, and 2 may be used to illustrate the Leading Coefficient Test.

Example 2 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of each polynomial function.

a. $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

Solution

a. Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 3.19.

b. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 3.20.

c. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 3.21.

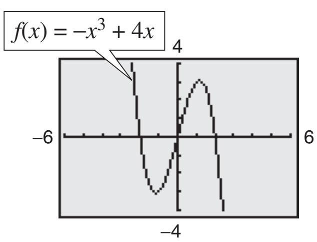


Figure 3.19

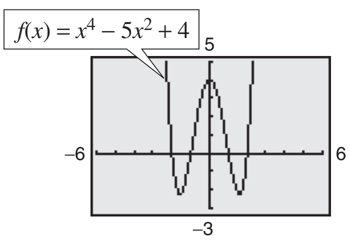


Figure 3.20

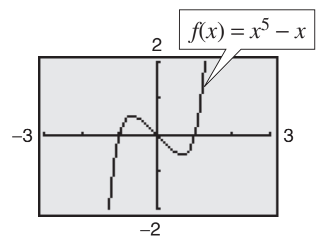


Figure 3.21

CHECKPOINT Now try Exercise 15.

A good test of students' understanding is to present a graph of a function without giving its equation, and ask the students what they can tell you about the function's degree and leading coefficient by looking at the graph. You might want to display a few such graphs on an overhead projector during class for practice.

In Example 2, note that the Leading Coefficient Test only tells you whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The function f has at most n real zeros. (You will study this result in detail in Section 3.4 on the Fundamental Theorem of Algebra.)
2. The graph of f has at most $n - 1$ relative **extrema** (relative **minima** or **maxima**).

Recall that a **zero** of a function f is a number x for which $f(x) = 0$. Finding the zeros of polynomial functions is one of the most important problems in algebra. You have already seen that there is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros. In other cases, you can use information about the zeros of a function to find a good viewing window.

Exploration

For each of the graphs in Example 2, count the number of zeros of the polynomial function and the number of relative extrema, and compare these numbers with the degree of the polynomial. What do you observe?

Additional Examples

Describe the right-hand and left-hand behavior of the graph of each function.

a. $f(x) = -x^4 + 2x^2 - 3x$
b. $f(x) = -x^5 + 3x^4 - x$
c. $f(x) = 2x^3 - 3x^2 + 5$

Solution

a. The graph falls to the left and right.
b. The graph rises to the left and falls to the right.
c. The graph falls to the left and rises to the right.

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Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

TECHNOLOGY SUPPORT

For instructions on how to use the *zero* or *root* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Finding zeros of polynomial functions is closely related to factoring and finding x-intercepts, as demonstrated in Examples 3, 4, and 5.

Example 3 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Algebraic Solution

$f(x) = x^3 - x^2 - 2x$ Write original function.

$0 = x^3 - x^2 - 2x$ Substitute 0 for $f(x)$.

$0 = x(x^2 - x - 2)$ Remove common monomial factor.

$0 = x(x - 2)(x + 1)$ Factor completely.

So, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$.

Check

$(0)^3 - (0)^2 - 2(0) = 0$ $x = 0$ is a zero. ✓

$(2)^3 - (2)^2 - 2(2) = 0$ $x = 2$ is a zero. ✓

$(-1)^3 - (-1)^2 - 2(-1) = 0$ $x = -1$ is a zero. ✓

CHECKPOINT Now try Exercise 33.

Graphical Solution

Use a graphing utility to graph $y = x^3 - x^2 - 2x$. In Figure 3.22, the graph appears to have the x-intercepts $(0, 0)$, $(2, 0)$, and $(-1, 0)$. Use the *zero* or *root* feature, or the *zoom* and *trace* features, of the graphing utility to verify these intercepts. Note that this third-degree polynomial has two relative extrema, at $(-0.55, 0.63)$ and $(1.22, -2.11)$.

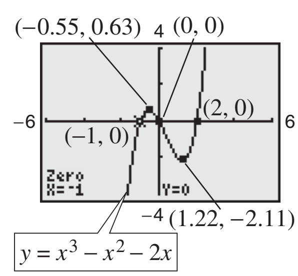

Figure 3.22

Example 4 Analyzing a Polynomial Function

Find all real zeros and relative extrema of $f(x) = -2x^4 + 2x^2$.

Solution

$0 = -2x^4 + 2x^2$ Substitute 0 for $f(x)$.

$0 = -2x^2(x^2 - 1)$ Remove common monomial factor.

$0 = -2x^2(x - 1)(x + 1)$ Factor completely.

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 3.23. Using the *minimum* and *maximum* features of a graphing utility, you can approximate the three relative extrema to be $(-0.71, 0.5)$, $(0, 0)$, and $(0.71, 0.5)$.

CHECKPOINT Now try Exercise 45.

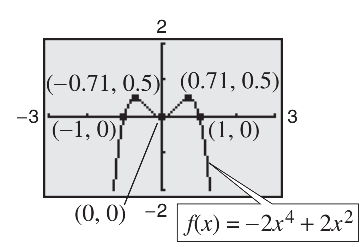

Figure 3.23

Repeated Zeros

For a polynomial function, a factor of $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. If k is odd, the graph *crosses* the x-axis at $x = a$.
2. If k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

STUDY TIP

In Example 4, note that because k is even, the factor $-2x^2$ yields the repeated zero $x = 0$. The graph touches (but does not cross) the x-axis at $x = 0$, as shown in Figure 3.23.

Example 5 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^5 - 3x^3 - x^2 - 4x - 1$.

Solution

Use a graphing utility to obtain the graph shown in Figure 3.24. From the graph, you can see that there are three zeros. Using the *zero* or *root* feature, you can determine that the zeros are approximately $x \approx -1.86$, $x \approx -0.25$, and $x \approx 2.11$. It should be noted that this fifth-degree polynomial factors as

$$f(x) = x^5 - 3x^3 - x^2 - 4x - 1 = (x^2 + 1)(x^3 - 4x - 1).$$

The three zeros obtained above are the zeros of the cubic factor $x^3 - 4x - 1$ (the quadratic factor $x^2 + 1$ has two complex zeros and so no *real* zeros).

CHECKPOINT Now try Exercise 47.

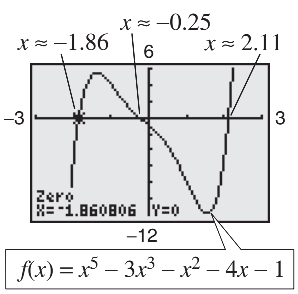

Figure 3.24

Example 6 Finding a Polynomial Function with Given Zeros

Find polynomial functions with the following zeros. (There are many correct solutions.)

a. $-\frac{1}{2}, 3, 3$ **b.** $3, 2 + \sqrt{11}, 2 - \sqrt{11}$

Prerequisite Skills

If you have difficulty with Example 6(b), review special products in Section P.3.

Solution

a. Note that the zero $x = -\frac{1}{2}$ corresponds to either $\left(x + \frac{1}{2}\right)$ or $(2x + 1)$. To avoid fractions, choose the second factor and write

$$\begin{aligned} f(x) &= (2x + 1)(x - 3)^2 \\ &= (2x + 1)(x^2 - 6x + 9) = 2x^3 - 11x^2 + 12x + 9. \end{aligned}$$

b. For each of the given zeros, form a corresponding factor and write

$$\begin{aligned} f(x) &= (x - 3)\left[x - \left(2 + \sqrt{11}\right)\right]\left[x - \left(2 - \sqrt{11}\right)\right] \\ &= (x - 3)\left[(x - 2) - \sqrt{11}\right]\left[(x - 2) + \sqrt{11}\right] \\ &= (x - 3)\left[(x - 2)^2 - \left(\sqrt{11}\right)^2\right] \\ &= (x - 3)(x^2 - 4x + 4 - 11) \\ &= (x - 3)(x^2 - 4x - 7) = x^3 - 7x^2 + 5x + 21. \end{aligned}$$

CHECKPOINT Now try Exercise 55.

Exploration

Use a graphing utility to graph

$$y_1 = x + 2$$
$$y_2 = (x + 2)(x - 1).$$

Predict the shape of the curve $y = (x + 2)(x - 1)(x - 3)$, and verify your answer with a graphing utility.

Note in Example 6 that there are many polynomial functions with the indicated zeros. In fact, multiplying the functions by any real number does not change the zeros of the function. For instance, multiply the function from part (b) by $\frac{1}{2}$ to obtain $f(x) = \frac{1}{2}x^3 - \frac{7}{2}x^2 + \frac{5}{2}x + \frac{21}{2}$. Then find the zeros of the function. You will obtain the zeros 3, $2 + \sqrt{11}$, and $2 - \sqrt{11}$, as given in Example 6.

Example 7 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$ by hand.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 3.25).
2. *Find the Real Zeros of the Polynomial.* By factoring

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4)$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 3) and $x = \frac{4}{3}$ (of odd multiplicity 1). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 3.25.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Be sure to choose points between the zeros and to the left and right of the zeros. Then plot the points (see Figure 3.26).

x	-1	0.5	1	1.5
$f(x)$	7	-0.31	-1	1.69

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.26. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$. If you are unsure of the shape of a portion of the graph, plot some additional points.

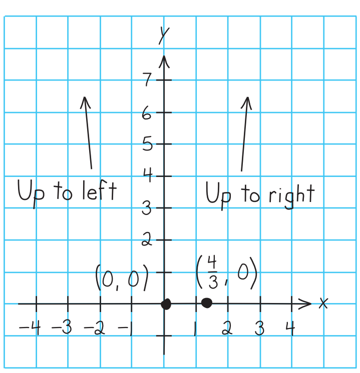

Figure 3.25

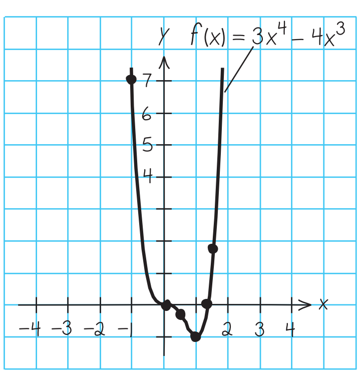

Figure 3.26

CHECKPOINT Now try Exercise 71.

TECHNOLOGY TIP

It is easy to make mistakes when entering functions into a graphing utility. So, it is important to have an understanding of the basic shapes of graphs and to be able to graph simple polynomials *by hand*. For example, suppose you had entered the function in Example 7 as $y = 3x^5 - 4x^3$. By looking at the graph, what mathematical principles would alert you to the fact that you had made a mistake?

Exploration

Partner Activity Multiply three, four, or five distinct linear factors to obtain the equation of a polynomial function of degree 3, 4, or 5. Exchange equations with your partner and sketch, *by hand*, the graph of the equation that your partner wrote. When you are finished, use a graphing utility to check each other's work.

Activities

1. Find all of the real zeros of $f(x) = 6x^4 - 33x^3 - 18x^2$.
   Answer: $-\frac{1}{2}, 0, 6$
2. Determine the right-hand and left-hand behavior of $f(x) = 6x^4 - 33x^3 - 18x^2$.
   Answer: The graph rises to the left and right.
3. Find a polynomial function of degree 3 that has zeros of 0, 2, and $-\frac{1}{3}$.
   Answer: $f(x) = 3x^3 - 5x^2 - 2x$

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Example 8 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 3.27).
2. *Find the Real Zeros of the Polynomial.* By factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

   you can see that the real zeros of f are $x = 0$ (of odd multiplicity 1) and $x = \frac{3}{2}$ (of even multiplicity 2). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 3.27.
3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points (see Figure 3.28.)

x	-0.5	0.5	1	2
$f(x)$	4	-1	-0.5	-1

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.28. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ and touches (but does not cross) the x-axis at $\left(\frac{3}{2}, 0\right)$.

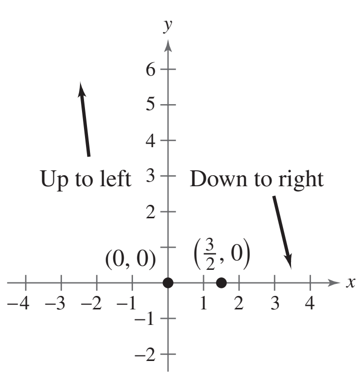


Figure 3.27

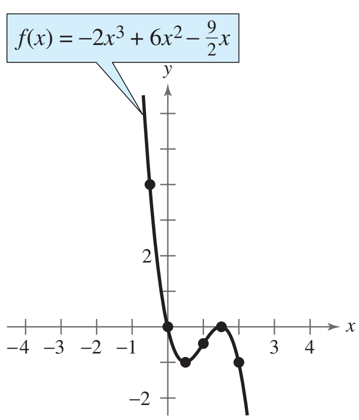


Figure 3.28

CHECKPOINT Now try Exercise 73.

TECHNOLOGY TIP Remember that when using a graphing utility to verify your graphs, you may need to adjust your viewing window in order to see all the features of the graph.

STUDY TIP

Observe in Example 8 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if a zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side of the zero to the other side. If a zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side of the zero to the other side. The following table helps to illustrate this result.

x	-0.5	0	0.5
$f(x)$	4	0	-1
Sign	$+$		$-$

x	1	$\frac{3}{2}$	2
$f(x)$	-0.5	0	-1
Sign	$-$		$-$

This sign analysis may be helpful in graphing polynomial functions.

The Intermediate Value Theorem

The **Intermediate Value Theorem** concerns the existence of real zeros of polynomial functions. The theorem states that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 3.29.)

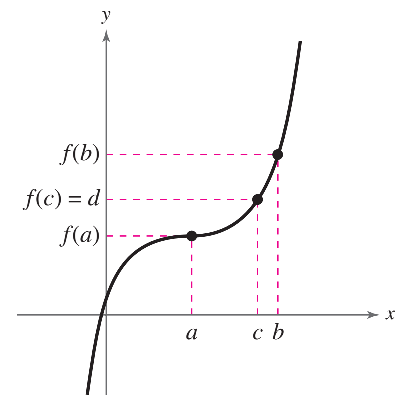


Figure 3.29

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

This theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, you can conclude that the function has at least one real zero between these two values. For example, the function $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1.

Example 9 Approximating the Zeros of a Function

Find three intervals of length 1 in which the polynomial $f(x) = 12x^3 - 32x^2 + 3x + 5$ is guaranteed to have a zero.

Graphical Solution

Use a graphing utility to graph

$$y = 12x^3 - 32x^2 + 3x + 5$$

as shown in Figure 3.30.

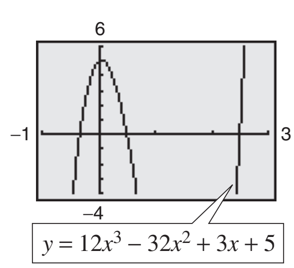


Figure 3.30

From the figure, you can see that the graph crosses the x-axis three times—between -1 and 0, between 0 and 1, and between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

CHECKPOINT Now try Exercise 79.

Numerical Solution

Use the *table* feature of a graphing utility to create a table of function values. Scroll through the table looking for consecutive function values that differ in sign. For instance, from the table in Figure 3.31 you can see that $f(-1)$ and $f(0)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between -1 and 0. Similarly, $f(0)$ and $f(1)$ differ in sign, so the function has a zero between 0 and 1. Likewise, $f(2)$ and $f(3)$ differ in sign, so the function has a zero between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

X	Y1
-2	-225
-1	-42
0	5
1	-12
2	-21
3	50
4	273

X= -1

Figure 3.31

3.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The graphs of all polynomial functions are _______ , which means that the graphs have no breaks, holes, or gaps.

2. The _______ is used to determine the left-hand and right-hand behavior of the graph of a polynomial function.

3. A polynomial function of degree n has at most _______ real zeros and at most _______ turning points, called _______ .

4. If $x = a$ is a zero of a polynomial function f, then the following statements are true.

(a) $x = a$ is a _______ of the polynomial equation $f(x) = 0$.

(b) _______ is a factor of the polynomial $f(x)$.

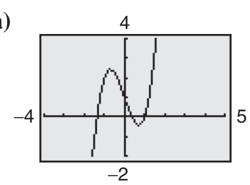
(c) $(a, 0)$ is an _______ of the graph of f.

5. If a zero of a polynomial function is of even multiplicity, then the graph of f _______ the x-axis, and if the zero is of odd multiplicity, then the graph of f _______ the x-axis.

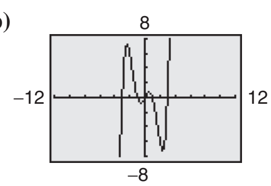
6. The _______ Theorem states that if f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

In Exercises 1–8, match the polynomial function with its graph. [The graphs are labeled (a) through (h).]

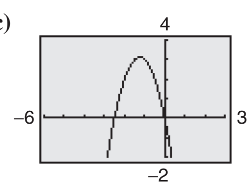
(a)



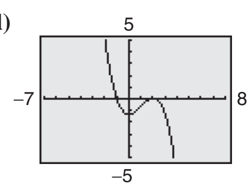
(b)



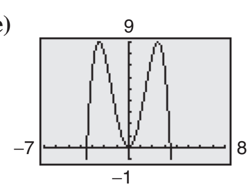
(c)



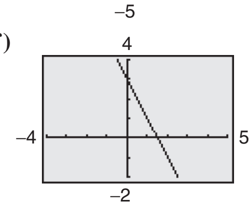
(d)



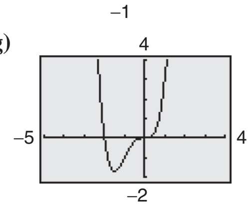
(e)



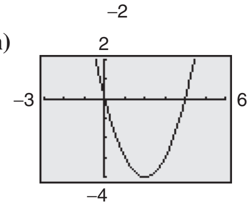
(f)



(g)



(h)



1. $f(x) = -2x + 3$

2. $f(x) = x^2 - 4x$

3. $f(x) = -2x^2 - 5x$

4. $f(x) = 2x^3 - 3x + 1$

5. $f(x) = -\frac{1}{4}x^4 + 3x^2$

6. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$

7. $f(x) = x^4 + 2x^3$

8. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

In Exercises 9 and 10, sketch the graph of $y = x^n$ and each specified transformation.

9. $y = x^3$

(a) $f(x) = (x - 2)^3$ (b) $f(x) = x^3 - 2$

(c) $f(x) = -\frac{1}{2}x^3$ (d) $f(x) = (x - 2)^3 - 2$

10. $y = x^4$

(a) $f(x) = (x + 5)^4$ (b) $f(x) = x^4 - 5$

(c) $f(x) = 4 - x^4$ (d) $f(x) = \frac{1}{2}(x - 1)^4$

Graphical Analysis **In Exercises 11–14, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough so that the right-hand and left-hand behaviors of f and g appear identical. Show both graphs.**

11. $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$

12. $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$

13. $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$

14. $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

In Exercises 15–22, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function. Use a graphing utility to verify your result.

15. $f(x) = 2x^4 - 3x + 1$

16. $h(x) = 1 - x^6$

17. $g(x) = 5 - \frac{7}{2}x - 3x^2$

18. $f(x) = \frac{1}{3}x^3 + 5x$

19. $f(x) = \dfrac{6x^5 - 2x^4 + 4x^2 - 5x}{3}$

20. $f(x) = \dfrac{3x^7 - 2x^5 + 5x^3 + 6x^2}{4}$

21. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

22. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

In Exercises 23–32, find all the real zeros of the polynomial function. Determine the multiplicity of each zero. Use a graphing utility to verify your result.

23. $f(x) = x^2 - 25$

24. $f(x) = 49 - x^2$

25. $h(t) = t^2 - 6t + 9$

26. $f(x) = x^2 + 10x + 25$

27. $f(x) = x^2 + x - 2$

28. $f(x) = 2x^2 - 14x + 24$

29. $f(t) = t^3 - 4t^2 + 4t$

30. $f(x) = x^4 - x^3 - 20x^2$

31. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

32. $f(x) = \frac{5}{3}x^2 + \frac{8}{3}x - \frac{4}{3}$

Graphical Analysis **In Exercises 33–44, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those from part (a).**

33. $f(x) = 3x^2 - 12x + 3$

34. $g(x) = 5x^2 - 10x - 5$

35. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

36. $y = \frac{1}{4}x^3(x^2 - 9)$

37. $f(x) = x^5 + x^3 - 6x$

38. $g(t) = t^5 - 6t^3 + 9t$

39. $f(x) = 2x^4 - 2x^2 - 40$

40. $f(x) = 5x^4 + 15x^2 + 10$

41. $f(x) = x^3 - 4x^2 - 25x + 100$

42. $y = 4x^3 + 4x^2 - 7x + 2$

43. $y = 4x^3 - 20x^2 + 25x$

44. $y = x^5 - 5x^3 + 4x$

In Exercises 45–48, use a graphing utility to graph the function and approximate (accurate to three decimal places) any real zeros and relative extrema.

45. $f(x) = 2x^4 - 6x^2 + 1$

46. $f(x) = -\frac{3}{8}x^4 - x^3 + 2x^2 + 5$

47. $f(x) = x^5 + 3x^3 - x + 6$

48. $f(x) = -3x^3 - 4x^2 + x - 3$

In Exercises 49–58, find a polynomial function that has the given zeros. (There are many correct answers.)

49. $0, 4$

50. $-7, 2$

51. $0, -2, -3$

52. $0, 2, 5$

53. $4, -3, 3, 0$

54. $-2, -1, 0, 1, 2$

55. $1 + \sqrt{3}, 1 - \sqrt{3}$

56. $6 + \sqrt{3}, 6 - \sqrt{3}$

57. $2, 4 + \sqrt{5}, 4 - \sqrt{5}$

58. $4, 2 + \sqrt{7}, 2 - \sqrt{7}$

In Exercises 59–64, find a polynomial function with the given zeros, multiplicities, and degree. (There are many correct answers.)

59. Zero: -2, multiplicity: 2
Zero: -1, multiplicity: 1
Degree: 3

60. Zero: 3, multiplicity: 1
Zero: 2, multiplicity: 3
Degree: 4

61. Zero: -4, multiplicity: 2
Zero: 3, multiplicity: 2
Degree: 4

62. Zero: -5, multiplicity: 3
Zero: 0, multiplicity: 2
Degree: 5

63. Zero: -1, multiplicity: 2
Zero: -2, multiplicity: 1
Degree: 3
Rises to the left,
Falls to the right

64. Zero: -1, multiplicity: 2
Zero: 4, multiplicity: 2
Degree: 4
Falls to the left,
Falls to the right

In Exercises 65–68, sketch the graph of a polynomial function that satisfies the given conditions. If not possible, explain your reasoning. (There are many correct answers.)

65. Third-degree polynomial with two real zeros and a negative leading coefficient

66. Fourth-degree polynomial with three real zeros and a positive leading coefficient

67. Fifth-degree polynomial with three real zeros and a positive leading coefficient

68. Fourth-degree polynomial with two real zeros and a negative leading coefficient

In Exercises 69–78, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

69. $f(x) = x^3 - 9x$

70. $g(x) = x^4 - 4x^2$

71. $f(x) = x^3 - 3x^2$

72. $f(x) = 3x^3 - 24x^2$

73. $f(x) = -x^4 + 9x^2 - 20$

74. $f(x) = -x^6 + 7x^3 + 8$

75. $f(x) = x^3 + 3x^2 - 9x - 27$

76. $h(x) = x^5 - 4x^3 + 8x^2 - 32$

77. $g(t) = -\frac{1}{4}t^4 + 2t^2 - 4$

78. $g(x) = \frac{1}{10}(x^4 - 4x^3 - 2x^2 + 12x + 9)$

In Exercises 79–82, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero, and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your answers in part (a) by using the *table* feature of the graphing utility.

79. $f(x) = x^3 - 3x^2 + 3$

80. $f(x) = -2x^3 - 6x^2 + 3$

81. $g(x) = 3x^4 + 4x^3 - 3$

82. $h(x) = x^4 - 10x^2 + 2$

In Exercises 83–90, use a graphing utility to graph the function. Identify any symmetry with respect to the *x*-axis, *y*-axis, or origin. Determine the number of *x*-intercepts of the graph.

83. $f(x) = x^2(x + 6)$

84. $h(x) = x^3(x - 4)^2$

85. $g(t) = -\frac{1}{2}(t - 4)^2(t + 4)^2$

86. $g(x) = \frac{1}{8}(x + 1)^2(x - 3)^3$

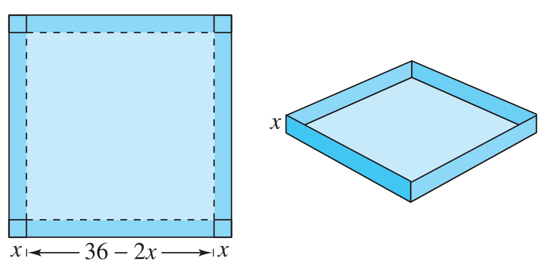
87. $f(x) = x^3 - 4x$

88. $f(x) = x^4 - 2x^2$

89. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

90. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

91. ***Numerical and Graphical Analysis*** An open box is to be made from a square piece of material 36 centimeters on a side by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).



(a) Verify that the volume of the box is given by the function $V(x) = x(36 - 2x)^2$.

(b) Determine the domain of the function V.

(c) Use the *table* feature of a graphing utility to create a table that shows various box heights x and the corresponding volumes V. Use the table to estimate a range of dimensions within which the maximum volume is produced.

(d) Use a graphing utility to graph V and use the range of dimensions from part (c) to find the x-value for which $V(x)$ is maximum.

92. ***Geometry*** An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is done by cutting equal squares from the corners and folding along the dashed lines, as shown in the figure.

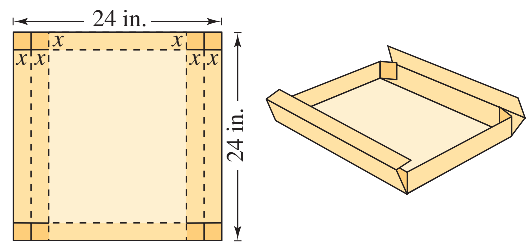


Figure for 92

(a) Verify that the volume of the box is given by the function $V(x) = 8x(6 - x)(12 - x)$.

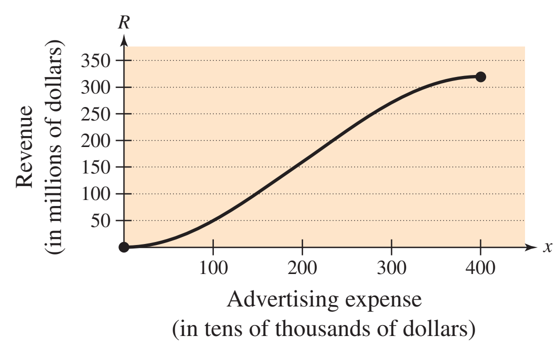
(b) Determine the domain of the function V.

(c) Sketch the graph of the function and estimate the value of x for which $V(x)$ is maximum.

93. ***Revenue*** The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = 0.00001(-x^3 + 600x^2),\ 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of the function shown in the figure to estimate the point on the graph at which the function is increasing most rapidly. This point is called the **point of diminishing returns** because any expense above this amount will yield less return per dollar invested in advertising.



94. ***Environment*** The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and t $(2 \le t \le 34)$ is its age (in years). Use a graphing utility to graph the function and estimate the age of the tree when it is growing most rapidly. This point is called the **point of diminishing returns** because the increase in growth will be less with each additional year. (*Hint:* Use a viewing window in which $0 \le x \le 35$ and $0 \le y \le 60$.)

Data Analysis **In Exercises 95–98, use the table, which shows the median prices (in thousands of dollars) of new privately owned U.S. homes in the Northeast y_1 and in the South y_2 for the years 1995 through 2004. The data can be approximated by the following models.**

$$y_1 = 0.3050t^3 - 6.949t^2 + 53.93t - 8.8$$

$$y_2 = 0.0330t^3 - 0.528t^2 + 8.35t + 65.2$$

In the models, t represents the year, with $t = 5$ corresponding to 1995. (Sources: National Association of Realtors)

Year, t	y_1	y_2
5	126.7	97.7
6	127.8	103.4
7	131.8	109.6
8	135.9	116.2
9	139.0	120.3
10	139.4	128.3
11	146.5	137.4
12	164.3	147.3
13	190.5	157.1
14	220.0	169.0

95. Use a graphing utility to plot the data and graph the model for y_1 in the same viewing window. How closely does the model represent the data?

96. Use a graphing utility to plot the data and graph the model for y_2 in the same viewing window. How closely does the model represent the data?

97. Use the models to predict the median prices of new privately owned homes in both regions in 2010. Do your answers seem reasonable? Explain.

98. Use the graphs of the models in Exercises 95 and 96 to write a short paragraph about the relationship between the median prices of homes in the two regions.

Synthesis

True or False? **In Exercises 99–104, determine whether the statement is true or false. Justify your answer.**

99. It is possible for a sixth-degree polynomial to have only one zero.

100. The graph of the function

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

rises to the left and falls to the right.

101. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ crosses the x-axis at $x = 1$.

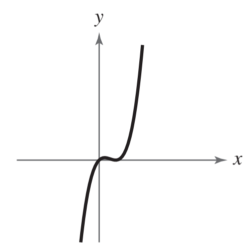
102. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ touches, but does not cross, the x-axis.

103. The graph of the function $f(x) = -x^2(x + 2)^3(x - 4)^2$ crosses the x-axis at $x = -2$.

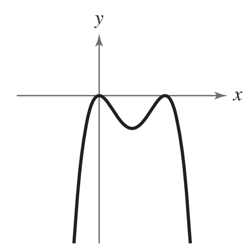
104. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ rises to the left and falls to the right.

Library of Parent Functions **In Exercises 105–107, determine which polynomial function(s) may be represented by the graph shown. There may be more than one correct answer.**

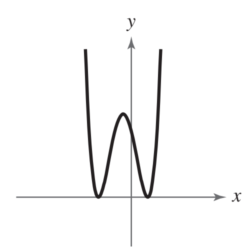
105. (a) $f(x) = x(x + 1)^2$
(b) $f(x) = x(x - 1)^2$
(c) $f(x) = x^2(x - 1)$
(d) $f(x) = -x(x + 1)^2$
(e) $f(x) = -x(x - 1)^2$



106. (a) $f(x) = x^2(x - 2)^2$
(b) $f(x) = x^2(x + 2)^2$
(c) $f(x) = x(x - 2)$
(d) $f(x) = -x^2(x + 2)^2$
(e) $f(x) = -x^2(x - 2)^2$



107. (a) $f(x) = (x - 1)^2(x + 2)^2$
(b) $f(x) = (x - 1)(x + 2)$
(c) $f(x) = (x + 1)^2(x - 2)^2$
(d) $f(x) = -(x - 1)^2(x + 2)^2$
(e) $f(x) = -(x + 1)^2(x - 2)^2$



Skills Review

In Exercises 108–113, let $f(x) = 14x - 3$ and $g(x) = 8x^2$. Find the indicated value.

108. $(f + g)(-4)$ **109.** $(g - f)(3)$

110. $(fg)\left(-\frac{4}{7}\right)$ **111.** $\left(\frac{f}{g}\right)(-1.5)$

112. $(f \circ g)(-1)$ **113.** $(g \circ f)(0)$

In Exercises 114–117, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.

114. $3(x - 5) < 4x - 7$ **115.** $2x^2 - x \geq 1$

116. $\frac{5x - 2}{x - 7} \leq 4$ **117.** $|x + 8| - 1 \geq 15$