

3.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

Consider the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice in Figure 3.32 that $x = 2$ appears to be a zero of f . Because $f(2) = 0$, you know that $x = 2$ is a zero of the polynomial function f , and that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that $f(x) = (x - 2) \cdot q(x)$. To find $q(x)$, you can use **long division of polynomials**.

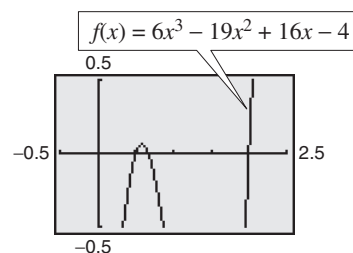


Figure 3.32

Example 1 Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

$$\begin{array}{r}
 \begin{array}{c} \text{Partial quotients} \\ \downarrow \quad \downarrow \quad \downarrow \\ 6x^2 - 7x + 2 \end{array} \\
 x - 2 \overline{) 6x^3 - 19x^2 + 16x - 4} \\
 \underline{6x^3 - 12x^2} \\
 - 7x^2 + 16x \\
 \underline{- 7x^2 + 14x} \\
 2x - 4 \\
 \underline{2x - 4} \\
 0
 \end{array}
 \quad
 \begin{array}{l}
 \text{Multiply: } 6x^2(x - 2). \\
 \text{Subtract.} \\
 \text{Multiply: } -7x(x - 2). \\
 \text{Subtract.} \\
 \text{Multiply: } 2(x - 2). \\
 \text{Subtract.}
 \end{array}$$

You can see that

$$\begin{aligned}
 6x^3 - 19x^2 + 16x - 4 &= (x - 2)(6x^2 - 7x + 2) \\
 &= (x - 2)(2x - 1)(3x - 2).
 \end{aligned}$$

Note that this factorization agrees with the graph of f (see Figure 3.32) in that the three x -intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

CHECKPOINT Now try Exercise 1.

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder and Factor Theorems.
- Use the Rational Zero Test to determine possible rational zeros of polynomial functions.
- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Why you should learn it

The Remainder Theorem can be used to determine the number of coal mines in the United States in a given year based on a polynomial model, as shown in Exercise 92 on page 290.



Ian Murphy/Getty Images

STUDY TIP

Note that in Example 1, the division process requires $-7x^2 + 14x$ to be subtracted from $-7x^2 + 16x$. Therefore it is implied that

$$\begin{array}{r}
 -7x^2 + 16x \\
 -(-7x^2 + 14x) \\
 \hline
 7x^2 - 14x
 \end{array}
 =
 \begin{array}{r}
 -7x^2 + 16x \\
 7x^2 - 14x
 \end{array}$$

and instead is written simply as

$$\begin{array}{r}
 -7x^2 + 16x \\
 -7x^2 + 14x \\
 \hline
 2x
 \end{array}$$

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, if you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$\begin{array}{r}
 \text{Divisor} \quad \rightarrow \quad x + 1 \overline{) x^2 + 3x + 5} \quad \leftarrow \text{Quotient} \\
 \phantom{x + 1 \overline{) }} \underline{x^2 + x} \quad \leftarrow \text{Dividend} \\
 \phantom{x + 1 \overline{) }} 2x + 5 \\
 \phantom{x + 1 \overline{) }} \underline{2x + 2} \\
 \phantom{x + 1 \overline{) }} 3 \quad \leftarrow \text{Remainder}
 \end{array}$$

In fractional form, you can write this result as follows.

$$\begin{array}{c}
 \text{Dividend} \\
 \hline
 x^2 + 3x + 5 \\
 \hline
 \text{Divisor} \\
 x + 1
 \end{array}
 =
 \begin{array}{c}
 \text{Quotient} \\
 \hline
 x + 2
 \end{array}
 +
 \begin{array}{c}
 \text{Remainder} \\
 \downarrow \\
 3 \\
 \hline
 \text{Divisor} \\
 x + 1
 \end{array}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \quad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm**.

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\begin{array}{c}
 f(x) = d(x)q(x) + r(x) \\
 \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
 \text{Dividend} \quad \text{Divisor} \quad \text{Quotient} \quad \text{Remainder}
 \end{array}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ divides evenly into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Note that one of the many uses of polynomial division is to write a function as a sum of terms in order to find slant asymptotes (see Section 3.6). This is a skill that is also used frequently in calculus.

Have students practice identifying the dividend, divisor, quotient, and remainder when dividing polynomials. For instance, in the division problem

$$\frac{x^3 - x + 1}{x - 1} = x^2 + x + \frac{1}{x - 1}$$

the dividend is $x^3 - x + 1$, the divisor is $x - 1$, the quotient is $x^2 + x$, and the remainder is 1.

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Example 2 Long Division of PolynomialsDivide $8x^3 - 1$ by $2x - 1$.**Solution**Because there is no x^2 -term or x -term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$\begin{array}{r}
 \overline{) 8x^3 + 0x^2 + 0x - 1} \\
 \underline{8x^3 - 4x^2} \\
 4x^2 + 0x \\
 \underline{4x^2 - 2x} \\
 2x - 1 \\
 \underline{2x - 1} \\
 0
 \end{array}$$

So, $2x - 1$ divides evenly into $8x^3 - 1$, and you can write

$$\frac{8x^3 - 1}{2x - 1} = 4x^2 + 2x + 1, \quad x \neq \frac{1}{2}.$$

CHECKPOINT Now try Exercise 7.

You can check the result of Example 2 by multiplying.

$$\begin{aligned}
 (2x - 1)(4x^2 + 2x + 1) &= 8x^3 + 4x^2 + 2x - 4x^2 - 2x - 1 \\
 &= 8x^3 - 1
 \end{aligned}$$

Example 3 Long Division of PolynomialsDivide $-2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$.**Solution**Begin by writing the dividend in descending powers of x .

$$\begin{array}{r}
 \overline{) 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
 \underline{2x^4 + 4x^3 - 6x^2} \\
 x^2 + 3x - 2 \\
 \underline{x^2 + 2x - 3} \\
 x + 1
 \end{array}$$

Remind students that when division yields a remainder, it is important that they write the remainder term correctly.

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

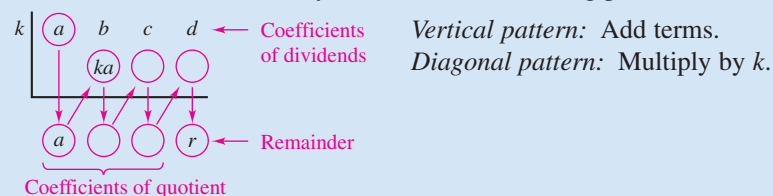
CHECKPOINT Now try Exercise 9.

Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form $x - k$. The shortcut is called **synthetic division**. The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



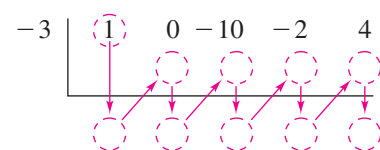
This algorithm for synthetic division works *only* for divisors of the form $x - k$. Remember that $x + k = x - (-k)$.

Example 4 Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

You should set up the array as follows. Note that a zero is included for each missing term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3 .

$$\begin{array}{r|rrrrr}
 -3 & 1 & 0 & -10 & -2 & 4 \\
 & & -3 & 9 & 3 & -3 \\
 \hline
 & 1 & -3 & -1 & 1 & 1
 \end{array}$$

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

Quotient: $x^3 - 3x^2 - x + 1$ Remainder: 1

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}$$

CHECKPOINT Now try Exercise 15.

Point out to students that they can use a graphing utility to check the answer to a polynomial division problem. When students graph both the original polynomial division problem and the answer in the same viewing window, the graphs should coincide.

Exploration

Evaluate the polynomial $x^4 - 10x^2 - 2x + 4$ at $x = -3$. What do you observe?

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem**.

The Remainder Theorem (See the proof on page 331.)

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

Example 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \quad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f . You can check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 = -9 \end{aligned}$$

 **CHECKPOINT** Now try Exercise 35.

Another important theorem is the **Factor Theorem**. This theorem states that you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

The Factor Theorem (See the proof on page 331.)

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Additional Example

Use the Remainder Theorem to evaluate $f(x) = 4x^2 - 10x - 21$ at $x = 5$.

Solution

Using synthetic division you obtain the following.

$$\begin{array}{r|rrr} 5 & 4 & -10 & -21 \\ & & 20 & 50 \\ \hline & 4 & 10 & 29 \end{array}$$

Because the remainder is 29, you can conclude that $f(5) = 29$.

Example 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array} \rightarrow \begin{array}{l} \text{0 remainder;} \\ (x - 2) \text{ is} \\ \text{a factor.} \end{array}$$

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array} \rightarrow \begin{array}{l} \text{0 remainder;} \\ (x + 3) \text{ is} \\ \text{a factor.} \end{array}$$

$2x^2 + 5x + 3$

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

CHECKPOINT Now try Exercise 45.

Graphical Solution

The graph of a polynomial with factors of $(x - 2)$ and $(x + 3)$ has x -intercepts at $x = 2$ and $x = -3$. Use a graphing utility to graph

$$y = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

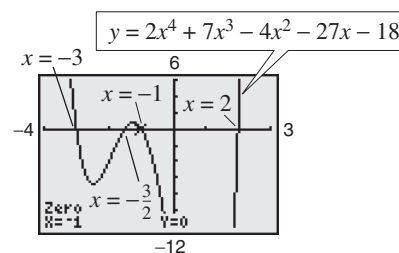


Figure 3.33

From Figure 3.33, you can see that the graph appears to cross the x -axis in two other places, near $x = -1$ and $x = -\frac{3}{2}$. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the other two intercepts to be $x = -1$ and $x = -\frac{3}{2}$. So, the factors of f are $(x - 2)$, $(x + 3)$, $(x + \frac{3}{2})$, and $(x + 1)$. You can rewrite the factor $(x + \frac{3}{2})$ as $(2x + 3)$, so the complete factorization of f is $f(x) = (x - 2)(x + 3)(2x + 3)(x + 1)$.

Using the Remainder in Synthetic Division

In summary, the remainder r , obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x -intercept of the graph of f .

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$, try sketching the graph of f . You should find that $(k, 0)$ is an x -intercept of the graph.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

has integer coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0 , and q is a factor of the leading coefficient a_n .

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Now that you have formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term. This case is illustrated in Example 7.

Example 7 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Possible rational zeros: ± 1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the polynomial has *no* rational zeros. Note from the graph of f in Figure 3.34 that f does have one real zero between -1 and 0 . However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

 **CHECKPOINT** Now try Exercise 49.

STUDY TIP

Graph the polynomial $y = x^3 - 53x^2 + 103x - 51$ in the standard viewing window. From the graph alone, it appears that there is only one zero. From the Leading Coefficient Test, you know that because the degree of the polynomial is odd and the leading coefficient is positive, the graph falls to the left and rises to the right. So, the function must have another zero. From the Rational Zero Test, you know that ± 51 might be zeros of the function. If you zoom out several times, you will see a more complete picture of the graph. Your graph should confirm that $x = 51$ is a zero of f .

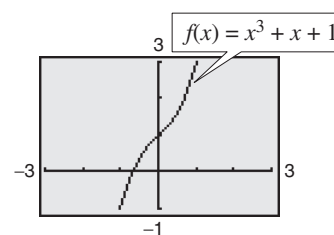


Figure 3.34

If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

1. A programmable calculator can be used to speed up the calculations.
2. A graphing utility can give a good estimate of the locations of the zeros.
3. The Intermediate Value Theorem, along with a table generated by a graphing utility, can give approximations of zeros.
4. The Factor Theorem and synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division.

Example 8 Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

Possible rational zeros:

$$\frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3) = (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in Figure 3.35.

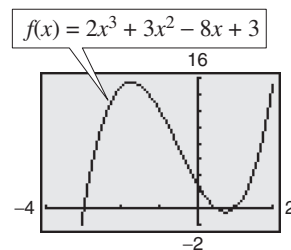


Figure 3.35

CHECKPOINT Now try Exercise 51.

A graphing utility can help you determine which possible rational zeros to test, as demonstrated in Example 9.

Activities

1. Use synthetic division to determine if $(x + 3)$ is a factor of $f(x) = 3x^3 + 4x^2 - 18x - 3$.

Answer: No

2. Divide using long division.

$$\frac{4x^5 - x^3 + 2x^2 - x}{2x + 1}$$

Answer:

$$2x^4 - x^3 + x - 1 + \frac{1}{2x + 1}$$

3. Use the Remainder Theorem to evaluate $f(-3)$ for $f(x) = 2x^3 - 4x^2 + 1$.

Answer: -89

4. Use the Rational Zero Test to find all the possible rational zeros of

$$f(x) = 6x^3 - x^2 + 9x + 4.$$

Answer: $\pm 1, \pm 2, \pm 4, \pm \frac{1}{6}, \pm \frac{1}{3}, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm \frac{4}{3}$

Example 9 Finding Real Zeros of a Polynomial Function

Find all the real zeros of $f(x) = 10x^3 - 15x^2 - 16x + 12$.

Solution

Because the leading coefficient is 10 and the constant term is 12, there is a long list of possible rational zeros.

Possible rational zeros:

$$\frac{\text{Factors of 12}}{\text{Factors of 10}} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

With so many possibilities (32, in fact), it is worth your time to use a graphing utility to focus on just a few. By using the *trace* feature of a graphing utility, it looks like three reasonable choices are $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$ (see Figure 3.36). Synthetic division shows that only $x = 2$ works. (You could also use the Factor Theorem to test these choices.)

$$\begin{array}{r|rrrr} 2 & 10 & -15 & -16 & 12 \\ & & 20 & 10 & -12 \\ \hline & 10 & 5 & -6 & 0 \end{array}$$

So, $x = 2$ is one zero and you have

$$f(x) = (x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.56 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.06$$

 **CHECKPOINT** Now try Exercise 55.

Other Tests for Zeros of Polynomials

You know that an n th-degree polynomial function can have *at most* n real zeros. Of course, many n th-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs**, sheds more light on the number of real zeros of a polynomial.

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

TECHNOLOGY TIP

You can use the *table* feature of a graphing utility to test the possible rational zeros of the function in Example 9, as shown below. Set the table to start at $x = -12$ and set the table step to 0.1. Look through the table to determine the values of x for which y_1 is 0.

X	Y ₁
1.7	-9.42
1.8	-7.08
1.9	-3.96
2.0	0
2.1	4.86
2.2	10.68
2.3	17.52

X=2

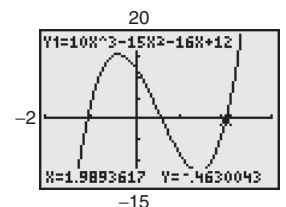


Figure 3.36

A **variation in sign** means that two consecutive (nonzero) coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 10 Using Descartes's Rule of Signs

Describe the possible real zeros of $f(x) = 3x^3 - 5x^2 + 6x - 4$.

Solution

The original polynomial has *three* variations in sign.

$$\begin{array}{cccc}
 & + & \text{to} & - \\
 & \downarrow & & \downarrow \\
 f(x) = & 3x^3 & - & 5x^2 & + & 6x & - & 4 \\
 & & & & \uparrow & & \uparrow & \\
 & & & & - & \text{to} & +
 \end{array}$$

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4 = -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. By using the *trace* feature of a graphing utility, you can see that the function has only one real zero (it is a positive number near $x = 1$), as shown in Figure 3.37.

 **CHECKPOINT** Now try Exercise 65.

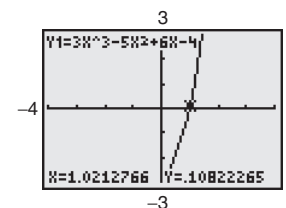


Figure 3.37

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound of the real zeros of f , which can help you eliminate possible real zeros. A real number b is an **upper bound** for the real zeros of f if no zeros are greater than b . Similarly, b is a **lower bound** if no real zeros of f are less than b .

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f .
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f .

Example 11 Finding the Zeros of a Polynomial Function

Find the real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of } 2}{\text{Factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$\begin{aligned} f(-x) &= 6(-x)^3 - 4(-x)^2 + 3(-x) - 2 \\ &= -6x^3 - 4x^2 - 3x - 2 \end{aligned}$$

has no variations in sign. As a result of these two findings, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative real zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. Therefore, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero.

 **CHECKPOINT** Now try Exercise 75.

Before concluding this section, here are two additional hints that can help you find the real zeros of a polynomial.

1. If the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. If you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, if you succeeded in writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x - 1)(x^2 - 4x - 1)$$

you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Exploration

Use a graphing utility to graph

$$y_1 = 6x^3 - 4x^2 + 3x - 2.$$

Notice that the graph intersects the x -axis at the point $(\frac{2}{3}, 0)$.

How does this information relate to the real zero found in Example 11? Use a graphing utility to graph

$$y_2 = x^4 - 5x^3 + 3x^2 + x.$$

How many times does the graph intersect the x -axis? How many real zeros does y_2 have?

Exploration

Use a graphing utility to graph

$$\begin{aligned} y &= x^3 + 4.9x^2 - 126x \\ &\quad + 382.5 \end{aligned}$$

in the standard viewing window. From the graph, what do the real zeros appear to be? Discuss how the mathematical tools of this section might help you realize that the graph does not show all the important features of the polynomial function. Now use the *zoom* feature to find all the zeros of this function.

3.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

1. Two forms of the Division Algorithm are shown below. Identify and label each part.

$$f(x) = d(x)q(x) + r(x) \quad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–7, fill in the blanks.

- The rational expression $p(x)/q(x)$ is called _____ if the degree of the numerator is greater than or equal to that of the denominator, and is called _____ if the degree of the numerator is less than that of the denominator.
- An alternative method to long division of polynomials is called _____, in which the divisor must be of the form $x - k$.
- The test that gives a list of the possible rational zeros of a polynomial function is known as the _____ Test.
- The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called _____ of _____.
- The _____ states that if a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.
- A real number b is an _____ for the real zeros of f if no zeros are greater than b , and is a _____ if no real zeros of f are less than b .

In Exercises 1–14, use long division to divide.

- Divide $2x^2 + 10x + 12$ by $x + 3$.
- Divide $5x^2 - 17x - 12$ by $x - 4$.
- Divide $x^4 + 5x^3 + 6x^2 - x - 2$ by $x + 2$.
- Divide $x^3 - 4x^2 - 17x + 6$ by $x - 3$.
- Divide $4x^3 - 7x^2 - 11x + 5$ by $4x + 5$.
- Divide $2x^3 - 3x^2 - 50x + 75$ by $2x - 3$.
- Divide $7x^3 + 3$ by $x + 2$.
- Divide $8x^4 - 5$ by $2x + 1$.
- $(x + 8 + 6x^3 + 10x^2) \div (2x^2 + 1)$
- $(1 + 3x^2 + x^4) \div (3 - 2x + x^2)$
- $(x^3 - 9) \div (x^2 + 1)$
- $(x^5 + 7) \div (x^3 - 1)$
- $\frac{2x^3 - 4x^2 - 15x + 5}{(x - 1)^2}$
- $\frac{x^4}{(x - 1)^3}$

In Exercises 15–24, use synthetic division to divide.

- $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$
- $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$
- $(6x^3 + 7x^2 - x + 26) \div (x - 3)$
- $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$
- $(9x^3 - 18x^2 - 16x + 32) \div (x - 2)$
- $(5x^3 + 6x + 8) \div (x + 2)$
- $(x^3 + 512) \div (x + 8)$
- $(x^3 - 729) \div (x - 9)$

$$23. \frac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}} \quad 24. \frac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$$

Graphical Analysis In Exercises 25–28, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify the results algebraically.

$$25. y_1 = \frac{x^2}{x + 2}, \quad y_2 = x - 2 + \frac{4}{x + 2}$$

$$26. y_1 = \frac{x^2 + 2x - 1}{x + 3}, \quad y_2 = x - 1 + \frac{2}{x + 3}$$

$$27. y_1 = \frac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \frac{39}{x^2 + 5}$$

$$28. y_1 = \frac{x^4 + x^2 - 1}{x^2 + 1}, \quad y_2 = x^2 - \frac{1}{x^2 + 1}$$

In Exercises 29–34, write the function in the form $f(x) = (x - k)q(x) + r(x)$ for the given value of k . Use a graphing utility to demonstrate that $f(k) = r$.

Function	Value of k
29. $f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
30. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$
31. $f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
32. $f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
33. $f(x) = 4x^3 - 6x^2 - 12x - 4$	$k = 1 - \sqrt{3}$
34. $f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

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In Exercises 35–38, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.

35. $f(x) = 2x^3 - 7x + 3$
 (a) $f(1)$ (b) $f(-2)$ (c) $f(\frac{1}{2})$ (d) $f(2)$
36. $g(x) = 2x^6 + 3x^4 - x^2 + 3$
 (a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$
37. $h(x) = x^3 - 5x^2 - 7x + 4$
 (a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$
38. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$
 (a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

In Exercises 39–42, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real zeros of the function.

Polynomial Equation	Value of x
39. $x^3 - 7x + 6 = 0$	$x = 2$
40. $x^3 - 28x - 48 = 0$	$x = -4$
41. $2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
42. $48x^3 - 80x^2 + 41x - 6 = 0$	$x = \frac{2}{3}$

In Exercises 43–48, (a) verify the given factors of the function f , (b) find the remaining factors of f , (c) use your results to write the complete factorization of f , and (d) list all real zeros of f . Confirm your results by using a graphing utility to graph the function.

Function	Factor(s)
43. $f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2)$
44. $f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3)$
45. $f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
46. $f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
47. $f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1)$
48. $f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1)$

In Exercises 49–52, use the Rational Zero Test to list all possible rational zeros of f . Then find the rational zeros.

49. $f(x) = x^3 + 3x^2 - x - 3$
50. $f(x) = x^3 - 4x^2 - 4x + 16$
51. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$
52. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$

Occasionally, throughout this text, you will be asked to round to a place value rather than to a number of decimal places.

In Exercises 53–60, find all real zeros of the polynomial function.

53. $f(z) = z^4 - z^3 - 2z - 4$
54. $f(x) = x^4 - x^3 - 29x^2 - x - 30$
55. $g(y) = 2y^4 + 7y^3 - 26y^2 + 23y - 6$
56. $h(x) = x^5 - x^4 - 3x^3 + 5x^2 - 2x$
57. $f(x) = 4x^4 - 55x^2 - 45x + 36$
58. $z(x) = 4x^4 - 43x^2 - 9x + 90$
59. $f(x) = 4x^5 + 12x^4 - 11x^3 - 42x^2 + 7x + 30$
60. $g(x) = 4x^5 + 8x^4 - 15x^3 - 23x^2 + 11x + 15$

Graphical Analysis In Exercises 61–64, (a) use the zero or root feature of a graphing utility to approximate (accurate to the nearest thousandth) the zeros of the function, (b) determine one of the exact zeros and use synthetic division to verify your result, and (c) factor the polynomial completely.

61. $h(t) = t^3 - 2t^2 - 7t + 2$
62. $f(s) = s^3 - 12s^2 + 40s - 24$
63. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$
64. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

In Exercises 65–68, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.

65. $f(x) = 2x^4 - x^3 + 6x^2 - x + 5$
66. $f(x) = 3x^4 + 5x^3 - 6x^2 + 8x - 3$
67. $g(x) = 4x^3 - 5x + 8$
68. $g(x) = 2x^3 - 4x^2 - 5$

In Exercises 69–74, (a) use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of f , (b) list the possible rational zeros of f , (c) use a graphing utility to graph f so that some of the possible zeros in parts (a) and (b) can be disregarded, and (d) determine all the real zeros of f .

69. $f(x) = x^3 + x^2 - 4x - 4$
70. $f(x) = -3x^3 + 20x^2 - 36x + 16$
71. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$
72. $f(x) = 4x^4 - 17x^2 + 4$
73. $f(x) = 32x^3 - 52x^2 + 17x + 3$
74. $f(x) = 4x^3 + 7x^2 - 11x - 18$

In Exercises 75–78, use synthetic division to verify the upper and lower bounds of the real zeros of f . Then find the real zeros of the function.

75. $f(x) = x^4 - 4x^3 + 15$

Upper bound: $x = 4$; Lower bound: $x = -1$

76. $f(x) = 2x^3 - 3x^2 - 12x + 8$

Upper bound: $x = 4$; Lower bound: $x = -3$

77. $f(x) = x^4 - 4x^3 + 16x - 16$

Upper bound: $x = 5$; Lower bound: $x = -3$

78. $f(x) = 2x^4 - 8x + 3$

Upper bound: $x = 3$; Lower bound: $x = -4$

In Exercises 79–82, find the rational zeros of the polynomial function.

79. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

80. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

81. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

82. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

In Exercises 83–86, match the cubic function with the correct number of rational and irrational zeros.

(a) Rational zeros: 0; Irrational zeros: 1

(b) Rational zeros: 3; Irrational zeros: 0

(c) Rational zeros: 1; Irrational zeros: 2

(d) Rational zeros: 1; Irrational zeros: 0

83. $f(x) = x^3 - 1$

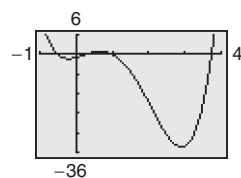
84. $f(x) = x^3 - 2$

85. $f(x) = x^3 - x$

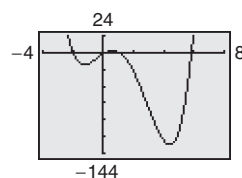
86. $f(x) = x^3 - 2x$

In Exercises 87–90, the graph of $y = f(x)$ is shown. Use the graph as an aid to find all the real zeros of the function.

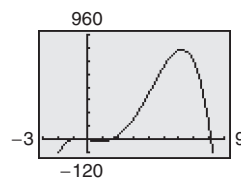
87. $y = 2x^4 - 9x^3 + 5x^2 + 3x - 1$



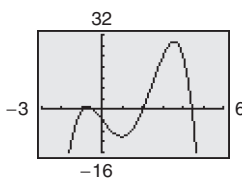
88. $y = x^4 - 5x^3 - 7x^2 + 13x - 2$



89. $y = -2x^4 + 17x^3 - 3x^2 - 25x - 3$



90. $y = -x^4 + 5x^3 - 10x - 4$



91. **U.S. Population** The table shows the populations P of the United States (in millions) from 1790 to 2000. (Source: U.S. Census Bureau)



Year	Population (in millions)
1790	3.9
1800	5.3
1810	7.2
1820	9.6
1830	12.9
1840	17.1
1850	23.2
1860	31.4
1870	39.8
1880	50.2
1890	63.0
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.2
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4

The population can be approximated by the equation

$$P = 0.0058t^3 + 0.500t^2 + 1.38t + 4.6, \quad -1 \leq t \leq 20$$

where t represents the year, with $t = -1$ corresponding to 1790, $t = 0$ corresponding to 1800, and so on.

- Use a graphing utility to graph the data and the equation in the same viewing window.
- How well does the model fit the data?
- Use the Remainder Theorem to evaluate the model for the year 2010. Do you believe this value is reasonable? Explain.

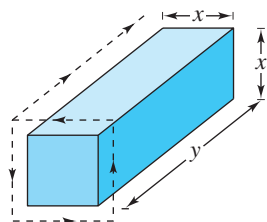
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92. Energy The number of coal mines C in the United States from 1980 to 2004 can be approximated by the equation $C = 0.232t^3 - 2.11t^2 - 261.8t + 5699$, for $0 \leq t \leq 24$, where t is the year, with $t = 0$ corresponding to 1980.

(Source: U.S. Energy Information Administration)

- Use a graphing utility to graph the model over the domain.
- Find the number of mines in 1980. Use the Remainder Theorem to find the number of mines in 1990.
- Could you use this model to predict the number of coal mines in the United States in the future? Explain.

93. Geometry A rectangular package sent by a delivery service can have a maximum combined length and girth (perimeter of a cross section) of 120 inches (see figure).



- Show that the volume of the package is given by the function $V(x) = 4x^2(30 - x)$.
- Use a graphing utility to graph the function and approximate the dimensions of the package that yield a maximum volume.
- Find values of x such that $V = 13,500$. Which of these values is a physical impossibility in the construction of the package? Explain.

94. Automobile Emissions The number of parts per million of nitric oxide emissions y from a car engine is approximated by the model $y = -5.05x^3 + 3,857x - 38,411.25$, for $13 \leq x \leq 18$, where x is the air-fuel ratio.

- Use a graphing utility to graph the model.
- It is observed from the graph that two air-fuel ratios produce 2400 parts per million of nitric oxide, with one being 15. Use the graph to approximate the second air-fuel ratio.
- Algebraically approximate the second air-fuel ratio that produces 2400 parts per million of nitric oxide. (*Hint:* Because you know that an air-fuel ratio of 15 produces the specified nitric oxide emission, you can use synthetic division.)

Synthesis

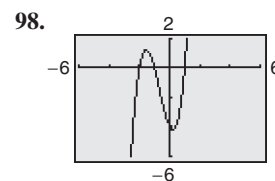
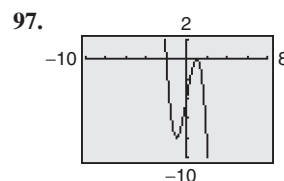
True or False? In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.

95. If $(7x + 4)$ is a factor of some polynomial function f , then $\frac{4}{7}$ is a zero of f .

96. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

Think About It In Exercises 97 and 98, the graph of a cubic polynomial function $y = f(x)$ with integer zeros is shown. Find the factored form of f .



99. Think About It Let $y = f(x)$ be a quartic polynomial with leading coefficient $a = -1$ and $f(\pm 1) = f(\pm 2) = 0$. Find the factored form of f .

100. Think About It Let $y = f(x)$ be a cubic polynomial with leading coefficient $a = 2$ and $f(-2) = f(1) = f(2) = 0$. Find the factored form of f .

101. Think About It Find the value of k such that $x - 4$ is a factor of $x^3 - kx^2 + 2kx - 8$.

102. Think About It Find the value of k such that $x - 3$ is a factor of $x^3 - kx^2 + 2kx - 12$.

103. Writing Complete each polynomial division. Write a brief description of the pattern that you obtain, and use your result to find a formula for the polynomial division $(x^n - 1)/(x - 1)$. Create a numerical example to test your formula.

(a) $\frac{x^2 - 1}{x - 1} = \square$

(b) $\frac{x^3 - 1}{x - 1} = \square$

(c) $\frac{x^4 - 1}{x - 1} = \square$

104. Writing Write a short paragraph explaining how you can check polynomial division. Give an example.

Skills Review

In Exercises 105–108, use any convenient method to solve the quadratic equation.

105. $9x^2 - 25 = 0$

106. $16x^2 - 21 = 0$

107. $2x^2 + 6x + 3 = 0$

108. $8x^2 - 22x + 15 = 0$

In Exercises 109–112, find a polynomial function that has the given zeros. (There are many correct answers.)

109. $0, -12$

110. $1, -3, 8$

111. $0, -1, 2, 5$

112. $2 + \sqrt{3}, 2 - \sqrt{3}$