3.5 Rational Functions and Asymptotes

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

Example 1 Finding the Domain of a Rational Function

Find the domain of $f(x) = 1/x$ and discuss the behavior of f near any excluded x-values.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

From the table, note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. Because $f(x)$ decreases without bound from the left and increases without bound from the right, you can conclude that f is not continuous. The graph of f is shown in Figure 3.42.

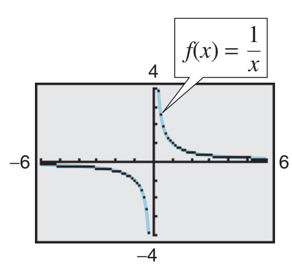

Figure 3.42

CHECKPOINT Now try Exercise 1.

What you should learn

- Find the domains of rational functions.
- Find horizontal and vertical asymptotes of graphs of rational functions.
- Use rational functions to model and solve real-life problems.

Why you should learn it

Rational functions are convenient in modeling a wide variety of real-life problems, such as environmental scenarios. For instance, Exercise 40 on page 306 shows how to determine the cost of recycling bins in a pilot project.

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Exploration

Use the *table* and *trace* features of a graphing utility to verify that the function $f(x) = 1/x$ in Example 1 is not continuous.

TECHNOLOGY TIP

The graphing utility graphs in this section and the next section were created using the *dot* mode. A blue curve is placed behind the graphing utility's display to indicate where the graph should appear. You will learn more about how graphing utilities graph rational functions in the next section.

Library of Parent Functions: Rational Function

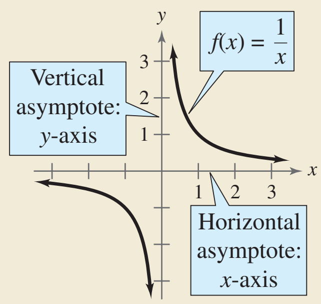
A *rational function* $f(x)$ is the quotient of two polynomials,

$$f(x) = \frac{N(x)}{D(x)}.$$

A rational function is not defined at values of x for which $D(x) = 0$. Near these values the graph of the rational function may increase or decrease without bound. The simplest type of rational function is the *reciprocal function* $f(x) = 1/x$. The basic characteristics of the reciprocal function are summarized below. A review of rational functions can be found in the *Study Capsules.*

Graph of $f(x) = \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
No intercepts
Decreasing on $(-\infty, 0)$ and $(0, \infty)$
Odd function
Origin symmetry
Vertical asymptote: y-axis
Horizontal asymptote: x-axis



Exploration

Use a table of values to determine whether the functions in Figure 3.43 are continuous. If the graph of a function has an asymptote, can you conclude that the function is not continuous? Explain.

Horizontal and Vertical Asymptotes

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \to -\infty \text{ as } x \to 0^-}_{f(x) \text{ decreases without bound as } x \text{ approaches 0 from the left.}} \qquad \underbrace{f(x) \to \infty \text{ as } x \to 0^+}_{f(x) \text{ increases without bound as } x \text{ approaches 0 from the right.}}$$

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in the figure above. The graph of f has a **horizontal asymptote**—the line $y = 0$. This means the values of $f(x) = 1/x$ approach zero as x increases or decreases without bound.

$$\underbrace{f(x) \to 0 \text{ as } x \to -\infty}_{f(x) \text{ approaches 0 as } x \text{ decreases without bound.}} \qquad \underbrace{f(x) \to 0 \text{ as } x \to \infty}_{f(x) \text{ approaches 0 as } x \text{ increases without bound.}}$$

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left.
2. The line $y = b$ is a **horizontal asymptote** of the graph of f if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.

Figure 3.43 shows the horizontal and vertical asymptotes of the graphs of three rational functions.

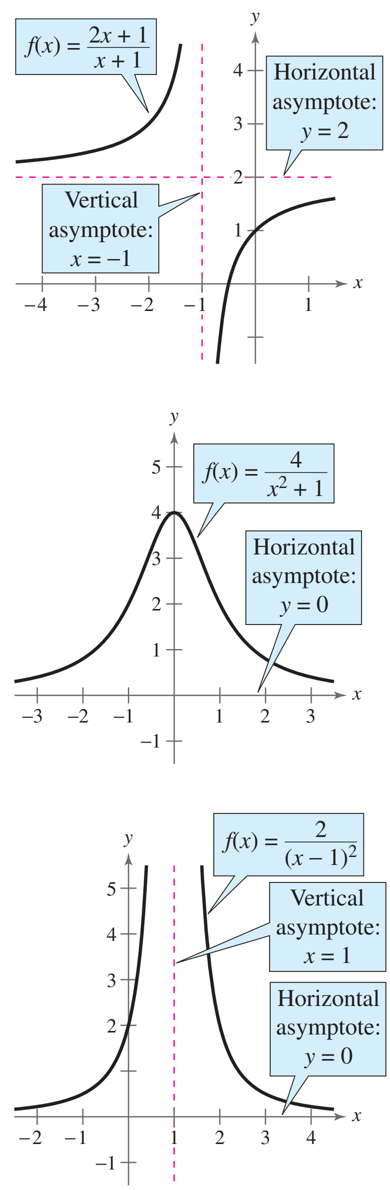


Figure 3.43

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.
2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.
   - **a.** If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   - **b.** If $n = m$, the graph of f has the line $y = a_n/b_m$ as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.
   - **c.** If $n > m$, the graph of f has no horizontal asymptote.

Exploration

Use a graphing utility to compare the graphs of y_1 and y_2.

$$y_1 = \frac{3x^3 - 5x^2 + 4x - 5}{2x^2 - 6x + 7}$$

$$y_2 = \frac{3x^3}{2x^2}$$

Start with a viewing window in which $-5 \le x \le 5$ and $-10 \le y \le 10$, then zoom out. Write a convincing argument that the shape of the graph of a rational function eventually behaves like the graph of $y = a_n x^n / b_m x^m$, where $a_n x^n$ is the leading term of the numerator and $b_m x^m$ is the leading term of the denominator.

Example 2 Finding Horizontal and Vertical Asymptotes

Find all horizontal and vertical asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

Solution

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$3x^2 + 1 = 0 \qquad \text{Set denominator equal to zero.}$$

Because this equation has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 3.44.

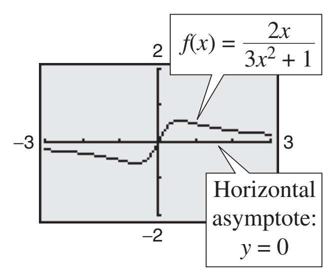


Figure 3.44

b. For this rational function, the degree of the numerator is *equal to* the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$x^2 - 1 = 0 \qquad \text{Set denominator equal to zero.}$$

$$(x + 1)(x - 1) = 0 \qquad \text{Factor.}$$

$$x + 1 = 0 \quad \Rightarrow \quad x = -1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 1 = 0 \quad \Rightarrow \quad x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes, as shown in Figure 3.45.

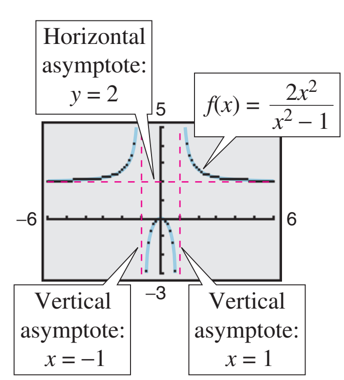


Figure 3.45

CHECKPOINT Now try Exercise 13.

Values for which a rational function is undefined (the denominator is zero) result in a vertical asymptote or a hole in the graph, as shown in Example 3.

Example 3 Finding Horizontal and Vertical Asymptotes and Holes

Find all horizontal and vertical asymptotes and holes in the graph of

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}.$$

Solution

For this rational function the degree of the numerator is *equal to* the degree of the denominator. The leading coefficients of the numerator and denominator are both 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote, as shown in Figure 3.46. To find any holes in the graph, note that the function is undefined at $x = -2$ and $x = 3$. Because $x = -2$ is not a vertical asymptote of the function, there is a hole in the graph at $x = -2$. To find the y-coordinate of the hole, substitute $x = -2$ into the simplified form of the function.

$$y = \frac{x - 1}{x - 3} = \frac{-2 - 1}{-2 - 3} = \frac{3}{5}$$

So, the graph of the rational function has a hole at $\left(-2, \frac{3}{5}\right)$.

CHECKPOINT Now try Exercise 17.

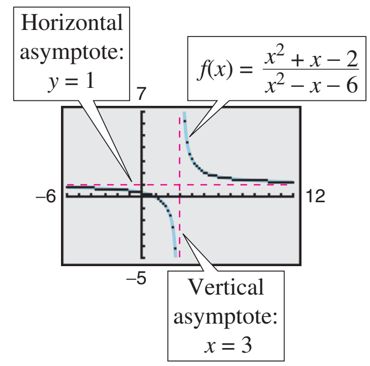
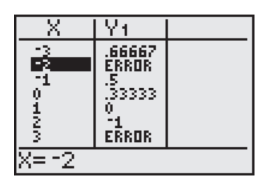


Figure 3.46

TECHNOLOGY TIP

Notice in Figure 3.46 that the function appears to be defined at $x = -2$. Because the domain of the function is all real numbers except $x = -2$ and $x = 3$, you know this is not true. Graphing utilities are limited in their resolution and therefore may not show a break or hole in the graph. Using the *table* feature of a graphing utility, you can verify that the function is not defined at $x = -2$.



X	Y1
-3	.66667
-2	ERROR
-1	.5
0	.33333
1	0
2	-1
3	ERROR

X=-2

Example 4 Finding a Function's Domain and Asymptotes

For the function f, find (a) the domain of f, (b) the vertical asymptote of f, and (c) the horizontal asymptote of f.

$$f(x) = \frac{3x^3 + 7x^2 + 2}{-4x^3 + 5}$$

Solution

a. Because the denominator is zero when $-4x^3 + 5 = 0$, solve this equation to determine that the domain of f is all real numbers except $x = \sqrt[3]{\frac{5}{4}}$.

b. Because the denominator of f has a zero at $x = \sqrt[3]{\frac{5}{4}}$, and $\sqrt[3]{\frac{5}{4}}$ is not a zero of the numerator, the graph of f has the vertical asymptote $x = \sqrt[3]{\frac{5}{4}} \approx 1.08$.

c. Because the degrees of the numerator and denominator are the same, and the leading coefficient of the numerator is 3 and the leading coefficient of the denominator is -4, the horizontal asymptote of f is $y = -\frac{3}{4}$.

CHECKPOINT Now try Exercise 19.

Example 5 A Graph with Two Horizontal Asymptotes

A function that is not rational can have two horizontal asymptotes—one to the left and one to the right. For instance, the graph of

$$f(x) = \frac{x + 10}{|x| + 2}$$

is shown in Figure 3.47. It has the line $y = -1$ as a horizontal asymptote to the left and the line $y = 1$ as a horizontal asymptote to the right. You can confirm this by rewriting the function as follows.

$$f(x) = \begin{cases} \dfrac{x + 10}{-x + 2}, & x < 0 \qquad |x| = -x \text{ for } x < 0 \\[2ex] \dfrac{x + 10}{x + 2}, & x \ge 0 \qquad |x| = x \text{ for } x \ge 0 \end{cases}$$

CHECKPOINT Now try Exercise 21.

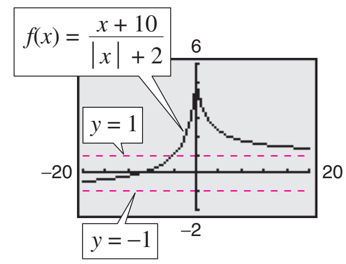


Figure 3.47

Applications

There are many examples of asymptotic behavior in real life. For instance, Example 6 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

Exploration

The *table* feature of a graphing utility can be used to estimate vertical and horizontal asymptotes of rational functions. Use the *table* feature to find any horizontal or vertical asymptotes of

$$f(x) = \frac{2x}{x + 1}.$$

Write a statement explaining how you found the asymptote(s) using the table.

Example 6 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing $p\%$ of the smokestack pollutants is given by $C = 80{,}000p/(100 - p)$ for $0 \le p < 100$. Use a graphing utility to graph this function. You are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

Solution

The graph of this function is shown in Figure 3.48. Note that the graph has a vertical asymptote at $p = 100$. Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \approx \$453{,}333. \qquad \text{Evaluate } C \text{ at } p = 85.$$

If the new law increases the percent removal to 90%, the cost will be

$$C = \frac{80{,}000(90)}{100 - 90} = \$720{,}000. \qquad \text{Evaluate } C \text{ at } p = 90.$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \text{Subtract 85\% removal cost from 90\% removal cost.}$$

CHECKPOINT Now try Exercise 39.

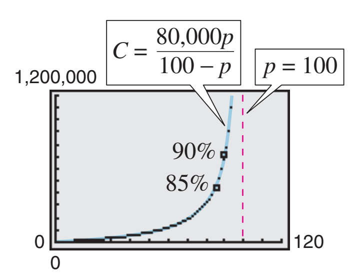


Figure 3.48

Example 7 Ultraviolet Radiation

For a person with sensitive skin, the amount of time T (in hours) the person can be exposed to the sun with minimal burning can be modeled by

$$T = \frac{0.37s + 23.8}{s}, \quad 0 < s \leq 120$$

where s is the Sunsor Scale reading. The Sunsor Scale is based on the level of intensity of UVB rays. (Source: Sunsor, Inc.)

a. Find the amounts of time a person with sensitive skin can be exposed to the sun with minimal burning when $s = 10$, $s = 25$, and $s = 100$.

b. If the model were valid for all $s > 0$, what would be the horizontal asymptote of this function, and what would it represent?

TECHNOLOGY SUPPORT

For instructions on how to use the *value* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Algebraic Solution

a. When $s = 10$, $T = \dfrac{0.37(10) + 23.8}{10}$

$= 2.75$ hours.

When $s = 25$, $T = \dfrac{0.37(25) + 23.8}{25}$

≈ 1.32 hours.

When $s = 100$, $T = \dfrac{0.37(100) + 23.8}{100}$

≈ 0.61 hour.

b. Because the degrees of the numerator and denominator are the same for

$$T = \frac{0.37s + 23.8}{s}$$

the horizontal asymptote is given by the ratio of the leading coefficients of the numerator and denominator. So, the graph has the line $T = 0.37$ as a horizontal asymptote. This line represents the shortest possible exposure time with minimal burning.

Graphical Solution

a. Use a graphing utility to graph the function

$$y_1 = \frac{0.37x + 23.8}{x}$$

using a viewing window similar to that shown in Figure 3.49. Then use the *trace* or *value* feature to approximate the values of y_1 when $x = 10$, $x = 25$, and $x = 100$. You should obtain the following values.

When $x = 10$, $y_1 = 2.75$ hours.

When $x = 25$, $y_1 \approx 1.32$ hours.

When $x = 100$, $y_1 \approx 0.61$ hour.

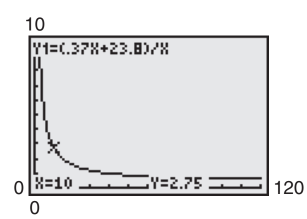


Figure 3.49

b. Continue to use the *trace* or *value* feature to approximate values of $f(x)$ for larger and larger values of x (see Figure 3.50). From this, you can estimate the horizontal asymptote to be $y = 0.37$. This line represents the shortest possible exposure time with minimal burning.

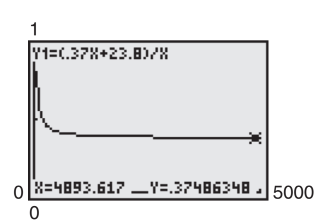


Figure 3.50

CHECKPOINT Now try Exercise 43.

3.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called _______ .

2. If $f(x) \to \pm\infty$ as $x \to a$ from the left (or right), then $x = a$ is a _______ of the graph of f.

3. If $f(x) \to b$ as $x \to \pm\infty$, then $y = b$ is a _______ of the graph of f.

In Exercises 1–6, (a) find the domain of the function, (b) complete each table, and (c) discuss the behavior of f near any excluded x-values.

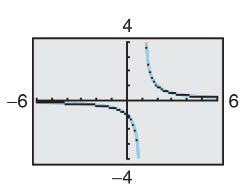
x	$f(x)$
0.5	
0.9	
0.99	
0.999	

x	$f(x)$
1.5	
1.1	
1.01	
1.001	

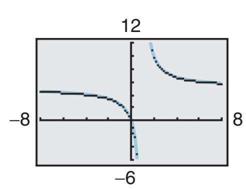
x	$f(x)$
5	
10	
100	
1000	

x	$f(x)$
-5	
-10	
-100	
-1000	

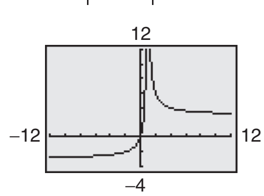
1. $f(x) = \dfrac{1}{x-1}$



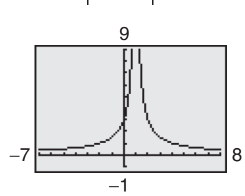
2. $f(x) = \dfrac{5x}{x-1}$



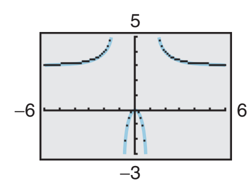
3. $f(x) = \dfrac{3x}{|x-1|}$



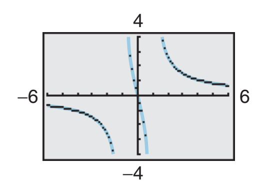
4. $f(x) = \dfrac{3}{|x-1|}$



5. $f(x) = \dfrac{3x^2}{x^2-1}$

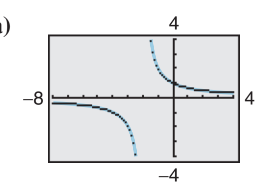


6. $f(x) = \dfrac{4x}{x^2-1}$

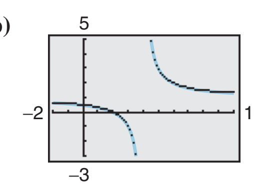


In Exercises 7–12, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

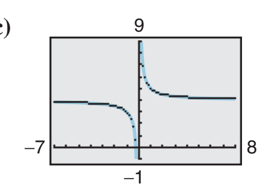
(a)



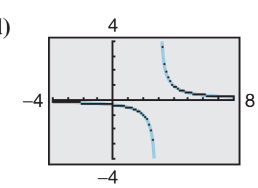
(b)



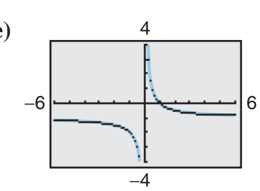
(c)



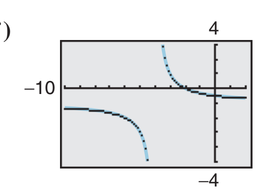
(d)



(e)



(f)



7. $f(x) = \dfrac{2}{x+2}$

8. $f(x) = \dfrac{1}{x-3}$

9. $f(x) = \dfrac{4x+1}{x}$

10. $f(x) = \dfrac{1-x}{x}$

11. $f(x) = \dfrac{x-2}{x-4}$

12. $f(x) = -\dfrac{x+2}{x+4}$

In Exercises 13–18, (a) identify any horizontal and vertical asymptotes and (b) identify any holes in the graph. Verify your answers numerically by creating a table of values.

13. $f(x) = \dfrac{1}{x^2}$

14. $f(x) = \dfrac{3}{(x-2)^3}$

15. $f(x) = \dfrac{x(2+x)}{2x - x^2}$

16. $f(x) = \dfrac{x^2 + 2x + 1}{2x^2 - x - 3}$

17. $f(x) = \dfrac{x^2 - 25}{x^2 + 5x}$

18. $f(x) = \dfrac{3 - 14x - 5x^2}{3 + 7x + 2x^2}$

In Exercises 19–22, (a) find the domain of the function, (b) decide if the function is continuous, and (c) identify any horizontal and vertical asymptotes. Verify your answer to part (a) both graphically by using a graphing utility and numerically by creating a table of values.

19. $f(x) = \dfrac{3x^2 + x - 5}{x^2 + 1}$

20. $f(x) = \dfrac{3x^2 + 1}{x^2 + x + 9}$

21. $f(x) = \dfrac{x - 3}{|x|}$

22. $f(x) = \dfrac{x + 1}{|x| + 1}$

Analytical and Numerical Explanation **In Exercises 23–26, (a) determine the domains of f and g, (b) simplify f and find any vertical asymptotes of f, (c) identify any holes in the graph of f, (d) complete the table, and (e) explain how the two functions differ.**

23. $f(x) = \dfrac{x^2 - 16}{x - 4}, \quad g(x) = x + 4$

x	1	2	3	4	5	6	7
$f(x)$							
$g(x)$							

24. $f(x) = \dfrac{x^2 - 9}{x - 3}, \quad g(x) = x + 3$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

25. $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}, \quad g(x) = \dfrac{x - 1}{x - 3}$

x	-2	-1	0	1	2	3	4
$f(x)$							
$g(x)$							

26. $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}, \quad g(x) = \dfrac{x + 2}{x - 1}$

x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							

Exploration **In Exercises 27–30, determine the value that the function f approaches as the magnitude of x increases. Is $f(x)$ greater than or less than this function value when x is positive and large in magnitude? What about when x is negative and large in magnitude?**

27. $f(x) = 4 - \dfrac{1}{x}$

28. $f(x) = 2 + \dfrac{1}{x - 3}$

29. $f(x) = \dfrac{2x - 1}{x - 3}$

30. $f(x) = \dfrac{2x - 1}{x^2 + 1}$

In Exercises 31–38, find the zeros (if any) of the rational function. Use a graphing utility to verify your answer.

31. $g(x) = \dfrac{x^2 - 4}{x + 3}$

32. $g(x) = \dfrac{x^3 - 8}{x^2 + 4}$

33. $f(x) = 1 - \dfrac{2}{x - 5}$

34. $h(x) = 5 + \dfrac{3}{x^2 + 1}$

35. $g(x) = \dfrac{x^2 - 2x - 3}{x^2 + 1}$

36. $g(x) = \dfrac{x^2 - 5x + 6}{x^2 + 4}$

37. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - 7x + 3}$

38. $f(x) = \dfrac{2x^2 + 3x - 2}{x^2 + x - 2}$

39. ***Environment*** The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by

$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of removing 10% of the pollutants.

(b) Find the cost of removing 40% of the pollutants.

(c) Find the cost of removing 75% of the pollutants.

(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to remove 100% of the pollutants? Explain.

40. ***Environment*** In a pilot project, a rural township is given recycling bins for separating and storing recyclable products. The cost C (in dollars) for supplying bins to $p\%$ of the population is given by

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of supplying bins to 15% of the population.

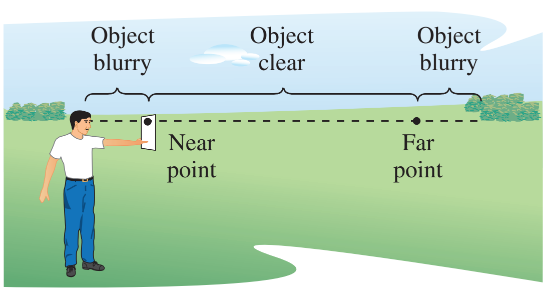
(b) Find the cost of supplying bins to 50% of the population.

(c) Find the cost of supplying bins to 90% of the population.

(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to supply bins to 100% of the residents? Explain.

41. ***Data Analysis*** The endpoints of the interval over which distinct vision is possible are called the *near point* and *far point* of the eye (see figure). With increasing age these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years).



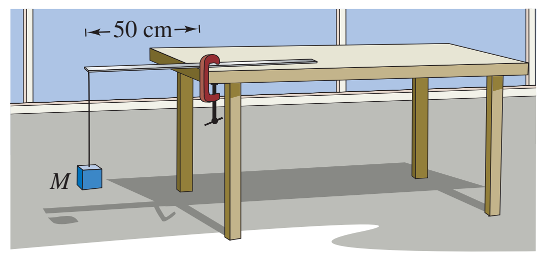
Age, x	**Near point, y**
16	3.0
32	4.7
44	9.8
50	19.7
60	39.4

(a) Find a rational model for the data. Take the reciprocals of the near points to generate the points $(x, 1/y)$. Use the *regression* feature of a graphing utility to find a linear model for the data. The resulting line has the form $1/y = ax + b$. Solve for y.

(b) Use the *table* feature of a graphing utility to create a table showing the predicted near point based on the model for each of the ages in the original table.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.

42. ***Data Analysis*** Consider a physics laboratory experiment designed to determine an unknown mass. A flexible metal meter stick is clamped to a table with 50 centimeters overhanging the edge (see figure). Known masses M ranging from 200 grams to 2000 grams are attached to the end of the meter stick. For each mass, the meter stick is displaced vertically and then allowed to oscillate. The average time t (in seconds) of one oscillation for each mass is recorded in the table.



Mass, M	**Time, t**
200	0.450
400	0.597
600	0.712
800	0.831
1000	0.906
1200	1.003
1400	1.088
1600	1.126
1800	1.218
2000	1.338

A model for the data is given by

$$t = \frac{38M + 16{,}965}{10(M + 5000)}.$$

(a) Use the *table* feature of a graphing utility to create a table showing the estimated time based on the model for each of the masses shown in the table. What can you conclude?

(b) Use the model to approximate the mass of an object when the average time for one oscillation is 1.056 seconds.

43. ***Wildlife*** The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is given by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \geq 0$$

where t is the time in years.

(a) Use a graphing utility to graph the model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases? Explain.

44. ***Defense*** The table shows the national defense outlays D (in billions of dollars) from 1997 to 2005. The data can be modeled by

$$D = \frac{1.493t^2 - 39.06t + 273.5}{0.0051t^2 - 0.1398t + 1}, \quad 7 \leq t \leq 15$$

where t is the year, with $t = 7$ corresponding to 1997. (Source: U.S. Office of Management and Budget)

Year	Defense outlays (in billions of dollars)
1997	270.5
1998	268.5
1999	274.9
2000	294.5
2001	305.5
2002	348.6
2003	404.9
2004	455.9
2005	465.9

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model represent the data?

(b) Use the model to predict the national defense outlays for the years 2010, 2015, and 2020. Are the predictions reasonable?

(c) Determine the horizontal asymptote of the graph of the model. What does it represent in the context of the situation?

Synthesis

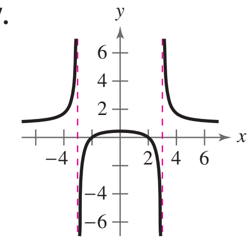
True or False? **In Exercises 45 and 46, determine whether the statement is true or false. Justify your answer.**

45. A rational function can have infinitely many vertical asymptotes.

46. A rational function must have at least one vertical asymptote.

Library of Parent Functions **In Exercises 47 and 48, identify the rational function represented by the graph.**

47.



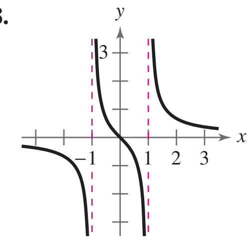
(a) $f(x) = \dfrac{x^2 - 9}{x^2 - 4}$

(b) $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$

(c) $f(x) = \dfrac{x - 4}{x^2 - 9}$

(d) $f(x) = \dfrac{x - 9}{x^2 - 4}$

48.



(a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(b) $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

(c) $f(x) = \dfrac{x}{x^2 - 1}$

(d) $f(x) = \dfrac{x}{x^2 + 1}$

Think About It **In Exercises 49–52, write a rational function f that has the specified characteristics. (There are many correct answers.)**

49. Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 0$
Zero: $x = 1$

50. Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 0$
Zero: $x = 2$

51. Vertical asymptotes: $x = -2, x = 1$
Horizontal asymptote: $y = 2$
Zeros: $x = 3, x = -3$

52. Vertical asymptotes: $x = -1, x = 2$
Horizontal asymptote: $y = -2$
Zeros: $x = -2, x = 3$

Skills Review

In Exercises 53–56, write the general form of the equation of the line that passes through the points.

53. $(3, 2), (0, -1)$

54. $(-6, 1), (4, -5)$

55. $(2, 7), (3, 10)$

56. $(0, 0), (-9, 4)$

In Exercises 57–60, divide using long division.

57. $(x^2 + 5x + 6) \div (x - 4)$

58. $(x^2 - 10x + 15) \div (x - 3)$

59. $(2x^4 + x^2 - 11) \div (x^2 + 5)$

60. $(4x^5 + 3x^3 - 10) \div (2x + 3)$