

P.3 Polynomials and Factoring

Polynomials

An algebraic expression is a collection of variables and real numbers. The most common type of algebraic expression is the **polynomial**. Some examples are

$$2x + 5, \quad 3x^4 - 7x^2 + 2x + 4, \quad \text{and} \quad 5x^2y^2 - xy + 3.$$

The first two are *polynomials in x* and the third is a *polynomial in x and y* . The terms of a polynomial in x have the form ax^k , where a is the **coefficient** and k is the **degree** of the term. For instance, the polynomial

$$2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$$

has coefficients 2, -5 , 0, and 1.

Definition of a Polynomial in x

Let $a_0, a_1, a_2, \dots, a_n$ be real numbers and let n be a nonnegative integer. A **polynomial in x** is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree n** , a_n is the **leading coefficient**, and a_0 is the **constant term**.

In **standard form**, a polynomial in x is written with descending powers of x . Polynomials with one, two, and three terms are called **monomials**, **binomials**, and **trinomials**, respectively.

A polynomial that has all zero coefficients is called the **zero polynomial**, denoted by 0. No degree is assigned to this particular polynomial. For polynomials in more than one variable, the degree of a *term* is the sum of the exponents of the variables in the term. The degree of the *polynomial* is the highest degree of its terms. For instance, the degree of the polynomial $-2x^3y^6 + 4xy - x^7y^4$ is 11 because the sum of the exponents in the last term is the greatest. Expressions such as the following are not polynomials.

$$x^3 - \sqrt{3x} = x^3 - (3x)^{1/2} \quad \text{The exponent } 1/2 \text{ is not an integer.}$$

$$x^2 + \frac{5}{x} = x^2 + 5x^{-1} \quad \text{The exponent } -1 \text{ is not a nonnegative integer.}$$

Example 1 Writing Polynomials in Standard Form

	Polynomial	Standard Form	Degree
a.	$4x^2 - 5x^7 - 2 + 3x$	$-5x^7 + 4x^2 + 3x - 2$	7
b.	$4 - 9x^2$	$-9x^2 + 4$	2
c.	8	$8 \ (8 = 8x^0)$	0

 **CHECKPOINT** Now try Exercise 15.

What you should learn

- Write polynomials in standard form.
- Add, subtract, and multiply polynomials.
- Use special products to multiply polynomials.
- Remove common factors from polynomials.
- Factor special polynomial forms.
- Factor trinomials as the product of two binomials.
- Factor by grouping.

Why you should learn it

Polynomials can be used to model and solve real-life problems. For instance, in Exercise 157 on page 34, a polynomial is used to model the total distance an automobile travels when stopping.



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STUDY TIP

Expressions are not polynomials if:

1. A variable is underneath a radical.
2. A polynomial expression (with degree greater than 0) is in the denominator of a term.

Operations with Polynomials

You can add and subtract polynomials in much the same way you add and subtract real numbers. Simply add or subtract the *like terms* (terms having exactly the same variables to exactly the same powers) by adding their coefficients. For instance, $-3xy^2$ and $5xy^2$ are like terms and their sum is

$$-3xy^2 + 5xy^2 = (-3 + 5)xy^2 = 2xy^2.$$

Example 2 Sums and Differences of Polynomials

Perform the indicated operation.

- a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$
 b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

Solution

$$\begin{aligned} \text{a. } & (5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8) \\ &= (5x^3 + x^3) + (-7x^2 + 2x^2) - x + (-3 + 8) \\ &= 6x^3 - 5x^2 - x + 5 \end{aligned}$$

Group like terms.

Combine like terms.

$$\begin{aligned} \text{b. } & (7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x) \\ &= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x \\ &= (7x^4 - 3x^4) + (-x^2 + 4x^2) + (-4x - 3x) + 2 \\ &= 4x^4 + 3x^2 - 7x + 2 \end{aligned}$$

Distributive Property

Group like terms.

Combine like terms.

 **CHECKPOINT** Now try Exercise 23.

To find the product of two polynomials, use the left and right Distributive Properties.

Example 3 Multiplying Polynomials: The FOIL Method

$$\begin{aligned} (3x - 2)(5x + 7) &= 3x(5x + 7) - 2(5x + 7) \\ &= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7) \\ &= 15x^2 + 21x - 10x - 14 \end{aligned}$$

Product of First terms	Product of Outer terms	Product of Inner terms	Product of Last terms
---------------------------	---------------------------	---------------------------	--------------------------

$$= 15x^2 + 11x - 14$$

Note that when using the **FOIL Method** (which can be used only to multiply two binomials), the outer (O) and inner (I) terms may be like terms that can be combined into one term.

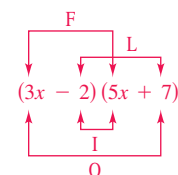
 **CHECKPOINT** Now try Exercise 39.

STUDY TIP

When a negative sign precedes an expression within parentheses, treat it like the coefficient (-1) and distribute the negative sign to each term inside the parentheses.

$$-(x^2 - x + 3) = -x^2 + x - 3$$

When using the FOIL method, the following scheme may be helpful.



Example 4 The Product of Two Trinomials

Find the product of $4x^2 + x - 2$ and $-x^2 + 3x + 5$.

Solution

When multiplying two polynomials, be sure to multiply *each* term of one polynomial by *each* term of the other. A vertical format is helpful.

$$\begin{array}{r}
 4x^2 + x - 2 \\
 \times \quad -x^2 + 3x + 5 \\
 \hline
 20x^2 + 5x - 10 \\
 12x^3 + 3x^2 - 6x \\
 -4x^4 - x^3 + 2x^2 \\
 \hline
 -4x^4 + 11x^3 + 25x^2 - x - 10
 \end{array}$$

Write in standard form.

Write in standard form.

← $5(4x^2 + x - 2)$

← $3x(4x^2 + x - 2)$

← $-x^2(4x^2 + x - 2)$

Combine like terms.

CHECKPOINT Now try Exercise 59.

Special Products**Special Products**

Let u and v be real numbers, variables, or algebraic expressions.

Special Product

Sum and Difference of Same Terms

$$(u + v)(u - v) = u^2 - v^2$$

Square of a Binomial

$$(u + v)^2 = u^2 + 2uv + v^2$$

$$(u - v)^2 = u^2 - 2uv + v^2$$

Cube of a Binomial

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

$$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$$

Example

$$(x + 4)(x - 4) = x^2 - 4^2 = x^2 - 16$$

$$(x + 3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 6x + 9$$

$$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2 = 9x^2 - 12x + 4$$

$$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$$

$$(x - 1)^3 = x^3 - 3x^2(1) + 3x(1^2) - 1^3 = x^3 - 3x^2 + 3x - 1$$

Example 5 The Product of Two Trinomials

Find the product of $x + y - 2$ and $x + y + 2$.

Solution

By grouping $x + y$ in parentheses, you can write the product of the trinomials as a special product.

$$\begin{aligned}
 (x + y - 2)(x + y + 2) &= [(x + y) - 2][(x + y) + 2] \\
 &= (x + y)^2 - 2^2 = x^2 + 2xy + y^2 - 4
 \end{aligned}$$

CHECKPOINT Now try Exercise 61.

To understand the individual patterns of special products, have students “derive” each product. Then explain how special products save time and how the pattern of the product must be recognized to factor an expression.

Factoring

The process of writing a polynomial as a product is called **factoring**. It is an important tool for solving equations and for simplifying rational expressions.

Unless noted otherwise, when you are asked to factor a polynomial, you can assume that you are looking for factors with integer coefficients. If a polynomial cannot be factored using integer coefficients, it is **prime** or **irreducible over the integers**. For instance, the polynomial $x^2 - 3$ is irreducible over the integers. Over the real numbers, this polynomial can be factored as

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).$$

A polynomial is **completely factored** when each of its factors is prime. So,

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4) \quad \text{Completely factored}$$

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4) \quad \text{Not completely factored}$$

is not completely factored. Its complete factorization is

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. The technique used here is the Distributive Property, $a(b + c) = ab + ac$, in the *reverse* direction. For instance, the polynomial $5x^2 + 15x$ can be factored as follows.

$$\begin{aligned} 5x^2 + 15x &= 5x(x) + 5x(3) && \text{5x is a common factor.} \\ &= 5x(x + 3) \end{aligned}$$

The first step in completely factoring a polynomial is to remove (factor out) any common factors, as shown in the next example.

Example 6 Removing Common Factors

Factor each expression.

a. $6x^3 - 4x$ b. $3x^4 + 9x^3 + 6x^2$ c. $(x - 2)(2x) + (x - 2)(3)$

Solution

a. $6x^3 - 4x = 2x(3x^2) - 2x(2) = 2x(3x^2 - 2)$ $2x$ is a common factor.

b. $3x^4 + 9x^3 + 6x^2 = 3x^2(x^2) + 3x^2(3x) + 3x^2(2)$ $3x^2$ is a common factor.
 $= 3x^2(x^2 + 3x + 2)$
 $= 3x^2(x + 1)(x + 2)$

c. $(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$ $x - 2$ is a common factor.

 **CHECKPOINT** Now try Exercise 73.

Factoring Special Polynomial Forms

Some polynomials have special forms that arise from the special product forms on page 26. You should learn to recognize these forms so that you can factor such polynomials easily.

Activities

1. Explain what happens when the parentheses are removed from the expression: $-(3x^2 - 4x + 2)$.

Answer: The sign of each term changes.
 $-3x^2 + 4x - 2$

2. Multiply using special products:

$$(\sqrt{2} + \sqrt{3})^2.$$

Answer: $5 + 2\sqrt{6}$

3. Multiply: $(2x + 1)(x - 5)$.

Answer: $2x^2 - 9x - 5$

Factoring Special Polynomial Forms

<i>Factored Form</i>	<i>Example</i>
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$

One of the easiest special polynomial forms to factor is the difference of two squares. Think of this form as follows.

$$u^2 - v^2 = (u + v)(u - v)$$

↑
↑

Difference
Opposite signs

To recognize perfect square terms, look for coefficients that are squares of integers and variables raised to *even powers*.

Example 7 Removing a Common Factor First

$$\begin{aligned}
 3 - 12x^2 &= 3(1 - 4x^2) && \text{3 is a common factor.} \\
 &= 3[1^2 - (2x)^2] && \text{Difference of two squares} \\
 &= 3(1 + 2x)(1 - 2x) && \text{Factored form}
 \end{aligned}$$

CHECKPOINT Now try Exercise 77.

Example 8 Factoring the Difference of Two Squares

$$\begin{aligned}
 \text{a. } (x + 2)^2 - y^2 &= [(x + 2) + y][(x + 2) - y] \\
 &= (x + 2 + y)(x + 2 - y) \\
 \text{b. } 16x^4 - 81 &= (4x^2)^2 - 9^2 && \text{Difference of two squares} \\
 &= (4x^2 + 9)(4x^2 - 9) \\
 &= (4x^2 + 9)[(2x)^2 - 3^2] && \text{Difference of two squares} \\
 &= (4x^2 + 9)(2x + 3)(2x - 3) && \text{Factored form}
 \end{aligned}$$

CHECKPOINT Now try Exercise 81.

STUDY TIP

In Example 7, note that the first step in factoring a polynomial is to check for a common factor. Once the common factor is removed, it is often possible to recognize patterns that were not immediately obvious.

A perfect square trinomial is the square of a binomial, as shown below.

$$u^2 + 2uv + v^2 = (u + v)^2 \quad \text{or} \quad u^2 - 2uv + v^2 = (u - v)^2$$

↑ Like signs ↑
↑ Like signs ↑

Note that the first and last terms are squares and the middle term is twice the product of u and v .

Example 9 Factoring Perfect Square Trinomials

Factor each trinomial.

a. $x^2 - 10x + 25$ b. $16x^2 + 8x + 1$

Solution

a. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2$ Rewrite in $u^2 - 2uv + v^2$ form.
 $= (x - 5)^2$

b. $16x^2 + 8x + 1 = (4x)^2 + 2(4x)(1) + 1^2$ Rewrite in $u^2 + 2uv + v^2$ form.
 $= (4x + 1)^2$

CHECKPOINT Now try Exercise 87.

The next two formulas show the sums and differences of cubes. Pay special attention to the signs of the terms.

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \quad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

↑ Like signs ↓
↑ Like signs ↓

↑ Unlike signs ↑
↑ Unlike signs ↑

Exploration

Rewrite $u^6 - v^6$ as the difference of two squares. Then find a formula for completely factoring $u^6 - v^6$. Use your formula to factor completely $x^6 - 1$ and $x^6 - 64$.

Example 10 Factoring the Difference of Cubes

Factor $x^3 - 27$.

Solution

$x^3 - 27 = x^3 - 3^3$ Rewrite 27 as 3^3 .
 $= (x - 3)(x^2 + 3x + 9)$ Factor.

CHECKPOINT Now try Exercise 92.

Example 11 Factoring the Sum of Cubes

$3x^3 + 192 = 3(x^3 + 64)$ 3 is a common factor.
 $= 3(x^3 + 4^3)$ Rewrite 64 as 4^3 .
 $= 3(x + 4)(x^2 - 4x + 16)$ Factor.

CHECKPOINT Now try Exercise 93.

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

$$ax^2 + bx + c = (\boxed{}x + \boxed{})(\boxed{}x + \boxed{})$$

Factors of a

Factors of c

The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx . For instance, in the trinomial $6x^2 + 17x + 5$, you can write all possible factorizations and determine which one has outer and inner products that add up to $17x$.

$$(6x + 5)(x + 1), (6x + 1)(x + 5), (2x + 1)(3x + 5), (2x + 5)(3x + 1)$$

You can see that $(2x + 5)(3x + 1)$ is the correct factorization because the outer (O) and inner (I) products add up to $17x$.

$$(2x + 5)(3x + 1) = 6x^2 + \overset{\text{F}}{2x} + \overset{\text{O}}{15x} + 5 = 6x^2 + \overset{\text{I}}{17x} + 5$$

Example 12 Factoring a Trinomial: Leading Coefficient Is 1

Factor $x^2 - 7x + 12$.

Solution

The possible factorizations are

$$(x - 2)(x - 6), (x - 1)(x - 12), \text{ and } (x - 3)(x - 4).$$

Testing the middle term, you will find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4). \quad \text{O} + \text{I} = -4x + (-3x) = -7x$$

 **CHECKPOINT** Now try Exercise 103.

Example 13 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor $2x^2 + x - 15$.

Solution

The eight possible factorizations are as follows.

$$(2x - 1)(x + 15), (2x + 1)(x - 15), (2x - 3)(x + 5), (2x + 3)(x - 5),$$

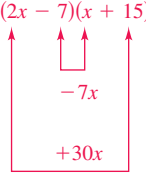
$$(2x - 5)(x + 3), (2x + 5)(x - 3), (2x - 15)(x + 1), (2x + 15)(x - 1)$$

Testing the middle term, you will find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3). \quad \text{O} + \text{I} = 6x - 5x = x$$

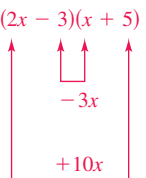
 **CHECKPOINT** Now try Exercise 111.

You can draw arrows to find the correct middle term. (Encourage students to find the middle term mentally.)

$$(2x - 7)(x + 15)$$


+30x

+23x Middle term

$$(2x - 3)(x + 5)$$


+10x

+7x Middle term

Point out to students that “testing the middle term” means using the FOIL method to find the sum of the outer and inner terms.

STUDY TIP

Factoring a trinomial can involve trial and error. However, once you have produced the factored form, it is an easy matter to check your answer. For instance, you can verify the factorization in Example 12 by multiplying out the expression $(x - 3)(x - 4)$ to see that you obtain the original trinomial, $x^2 - 7x + 12$.

Factoring by Grouping

Sometimes polynomials with more than three terms can be factored by a method called **factoring by grouping**.

Example 14 Factoring by Grouping

Use factoring by grouping to factor $x^3 - 2x^2 - 3x + 6$.

Solution

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor groups.} \\ &= (x - 2)(x^2 - 3) && (x - 2) \text{ is a common factor.} \end{aligned}$$

 **CHECKPOINT** Now try Exercise 115.

Factoring a trinomial can involve quite a bit of trial and error. Some of this trial and error can be lessened by using factoring by grouping. The key to this method of factoring is knowing how to rewrite the middle term. In general, to factor a trinomial $ax^2 + bx + c$ by grouping, choose factors of the product ac that add up to b and use these factors to rewrite the middle term.

Example 15 Factoring a Trinomial by Grouping

Use factoring by grouping to factor $2x^2 + 5x - 3$.

Solution

In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, which implies that the product ac is -6 . Now, because -6 factors as $(6)(-1)$ and $6 - 1 = 5 = b$, rewrite the middle term as $5x = 6x - x$. This produces the following.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.} \\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.} \\ &= 2x(x + 3) - (x + 3) && \text{Factor groups.} \\ &= (x + 3)(2x - 1) && (x + 3) \text{ is a common factor.} \end{aligned}$$

So, the trinomial factors as $2x^2 + 5x - 3 = (x + 3)(2x - 1)$.

 **CHECKPOINT** Now try Exercise 117.

STUDY TIP

When grouping terms be sure to strategically group terms that have a common factor.

Activities

1. Completely factor $4x^2 - 4x + 1$.
Answer: $(2x - 1)^2$
2. Completely factor $9x^4 - 144$.
Answer: $9(x^2 + 4)(x - 2)(x + 2)$

Guidelines for Factoring Polynomials

1. Factor out any common factors using the Distributive Property.
2. Factor according to one of the special polynomial forms.
3. Factor as $ax^2 + bx + c = (mx + r)(nx + s)$.
4. Factor by grouping.

P.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

- For the polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, the degree is _____ and the leading coefficient is _____.
- A polynomial that has all zero coefficients is called the _____.
- A polynomial with one term is called a _____.
- The letters in “FOIL” stand for the following. F _____ O _____ I _____ L _____
- If a polynomial cannot be factored using integer coefficients, it is called _____.
- The polynomial $u^2 + 2uv + v^2$ is called a _____.

In Exercises 1–6, match the polynomial with its description.
[The polynomials are labeled (a), (b), (c), (d), (e), and (f).]

- | | |
|---------------------------|---------------------------------|
| (a) $6x$ | (b) $1 - 4x^3$ |
| (c) $x^3 + 2x^2 - 4x + 1$ | (d) 7 |
| (e) $-3x^5 + 2x^3 + x$ | (f) $\frac{3}{4}x^4 + x^2 + 14$ |

- A polynomial of degree zero
- A trinomial of degree five
- A binomial with leading coefficient -4
- A monomial of positive degree
- A trinomial with leading coefficient $\frac{3}{4}$
- A third-degree polynomial with leading coefficient 1

In Exercises 7–10, write a polynomial that fits the description. (There are many correct answers.)

- A third-degree polynomial with leading coefficient -2
- A fifth-degree polynomial with leading coefficient 8
- A fourth-degree polynomial with a negative leading coefficient
- A third-degree trinomial with an even leading coefficient

In Exercises 11–16, write the polynomial in standard form. Then identify the degree and leading coefficient of the polynomial.

- | | |
|---------------------------|----------------------|
| 11. $3x + 4x^2 + 2$ | 12. $x^2 - 4 - 3x^4$ |
| 13. $5 - x^6$ | 14. $-13 + x^2$ |
| 15. $1 - x + 6x^4 - 2x^5$ | 16. $7 + 8x$ |

In Exercises 17–20, determine whether the expression is a polynomial. If so, write the polynomial in standard form.

- | | |
|------------------------|------------------------------|
| 17. $3x + 4x^3 - 5$ | 18. $5x^4 - 2x^2 + x^{-2}$ |
| 19. $\sqrt{x^2 - x^4}$ | 20. $\frac{x^2 + 2x - 3}{6}$ |

In Exercises 21–36, perform the operations and write the result in standard form.

- | | |
|--|----------------------------|
| 21. $(6x + 5) - (8x + 15)$ | |
| 22. $(2x^2 + 1) - (x^2 - 2x + 1)$ | |
| 23. $-(t^3 - 1) + (6t^3 - 5t)$ | |
| 24. $-(5x^2 - 1) - (-3x^2 + 5)$ | |
| 25. $(15x^2 - 6) - (-8.1x^3 - 14.7x^2 - 17)$ | |
| 26. $(15.6w - 14w - 17.4) - (16.9w^4 - 9.2w + 13)$ | |
| 27. $3x(x^2 - 2x + 1)$ | |
| 28. $y^2(4y^2 + 2y - 3)$ | |
| 29. $-5z(3z - 1)$ | |
| 30. $(-3x)(5x + 2)$ | |
| 31. $(1 - x^3)(4x)$ | 32. $-4x(3 - x^3)$ |
| 33. $(2.5x^2 + 5)(-3x)$ | 34. $(2 - 3.5y)(4y^3)$ |
| 35. $-2x(\frac{1}{8}x + 3)$ | 36. $6y(4 - \frac{3}{8}y)$ |

In Exercises 37–68, multiply or find the special product.

- | | |
|--|--|
| 37. $(x + 3)(x + 4)$ | 38. $(x - 5)(x + 10)$ |
| 39. $(3x - 5)(2x + 1)$ | 40. $(7x - 2)(4x - 3)$ |
| 41. $(2x - 5y)^2$ | 42. $(5 - 8x)^2$ |
| 43. $(x + 10)(x - 10)$ | |
| 44. $(2x + 3)(2x - 3)$ | |
| 45. $(x + 2y)(x - 2y)$ | |
| 46. $(4a + 5b)(4a - 5b)$ | |
| 47. $(2r^2 - 5)(2r^2 + 5)$ | |
| 48. $(3a^3 - 4b^2)(3a^3 + 4b^2)$ | |
| 49. $(x + 1)^3$ | 50. $(y - 4)^3$ |
| 51. $(2x - y)^3$ | 52. $(3x + 2y)^3$ |
| 53. $(\frac{1}{2}x - 5)^2$ | 54. $(\frac{2}{3}t + 4)^2$ |
| 55. $(\frac{1}{4}x - 3)(\frac{1}{4}x + 3)$ | 56. $(2x + \frac{1}{6})(2x - \frac{1}{6})$ |

57. $(2.4x + 3)^2$ 58. $(1.8y - 5)^2$
 59. $(-x^2 + x - 5)(3x^2 + 4x + 1)$
 60. $(x^2 + 3x + 2)(2x^2 - x + 4)$
 61. $[(x + 2z) + 5][(x + 2z) - 5]$
 62. $[(x - 3y) + z][(x - 3y) - z]$
 63. $[(x - 3) + y]^2$ 64. $[(x + 1) - y]^2$
 65. $5x(x + 1) - 3x(x + 1)$
 66. $(2x - 1)(x + 3) + 3(x + 3)$
 67. $(u + 2)(u - 2)(u^2 + 4)$
 68. $(x + y)(x - y)(x^2 + y^2)$

In Exercises 69–74, factor out the common factor.

69. $4x + 16$ 70. $5y - 30$
 71. $2x^3 - 6x$ 72. $3z^3 - 6z^2 + 9z$
 73. $3x(x - 5) + 8(x - 5)$
 74. $(5x - 4)^2 + (5x - 4)$

In Exercises 75–82, factor the difference of two squares.

75. $x^2 - 64$ 76. $x^2 - 81$
 77. $48y^2 - 27$ 78. $50 - 98z^2$
 79. $4x^2 - \frac{1}{9}$ 80. $\frac{25}{36}y^2 - 49$
 81. $(x - 1)^2 - 4$ 82. $25 - (z + 5)^2$

In Exercises 83–90, factor the perfect square trinomial.

83. $x^2 - 4x + 4$ 84. $x^2 + 10x + 25$
 85. $x^2 + x + \frac{1}{4}$ 86. $x^2 - \frac{4}{3}x + \frac{4}{9}$
 87. $4x^2 - 12x + 9$ 88. $25z^2 - 10z + 1$
 89. $4x^2 - \frac{4}{3}x + \frac{1}{9}$ 90. $9y^2 - \frac{3}{2}y + \frac{1}{16}$

In Exercises 91–100, factor the sum or difference of cubes.

91. $x^3 + 64$ 92. $x^3 - 1$
 93. $y^3 + 216$ 94. $z^3 - 125$
 95. $x^3 - \frac{8}{27}$ 96. $x^3 + \frac{8}{125}$
 97. $8x^3 - 1$ 98. $27x^3 + 8$
 99. $(x + 2)^3 - y^3$ 100. $(x - 3y)^3 - 8z^3$

In Exercises 101–114, factor the trinomial.

101. $x^2 + x - 2$ 102. $x^2 + 5x + 6$
 103. $s^2 - 5s + 6$ 104. $t^2 - t - 6$
 105. $20 - y - y^2$ 106. $24 + 5z - z^2$
 107. $3x^2 - 5x + 2$ 108. $3x^2 + 13x - 10$
 109. $2x^2 - x - 1$ 110. $2x^2 - x - 21$
 111. $5x^2 + 26x + 5$ 112. $8x^2 - 45x - 18$
 113. $-5u^2 - 13u + 6$ 114. $-6x^2 + 23x + 4$

Section P.3 Polynomials and Factoring

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In Exercises 115–120, factor by grouping.

115. $x^3 - x^2 + 2x - 2$
 116. $x^3 + 5x^2 - 5x - 25$
 117. $6x^2 + x - 2$
 118. $3x^2 + 10x + 8$
 119. $x^3 - 5x^2 + x - 5$
 120. $x^3 - x^2 + 3x - 3$

In Exercises 121–152, completely factor the expression.

121. $x^3 - 16x$ 122. $12x^2 - 48$
 123. $x^3 - x^2$ 124. $6x^2 - 54$
 125. $x^2 - 2x + 1$ 126. $9x^2 - 6x + 1$
 127. $1 - 4x + 4x^2$ 128. $16 - 6x - x^2$
 129. $2x^2 + 4x - 2x^3$ 130. $7y^2 + 15y - 2y^3$
 131. $9x^2 + 10x + 1$ 132. $13x + 6 + 5x^2$
 133. $\frac{1}{8}x^2 - \frac{1}{96}x - \frac{1}{16}$ 134. $\frac{1}{81}x^2 + \frac{2}{9}x - 8$
 135. $3x^3 + x^2 + 15x + 5$ 136. $5 - x + 5x^2 - x^3$
 137. $3u - 2u^2 + 6 - u^3$
 138. $x^4 - 4x^3 + x^2 - 4x$
 139. $2x^3 + x^2 - 8x - 4$
 140. $3x^3 + x^2 - 27x - 9$
 141. $(x^2 + 1)^2 - 4x^2$
 142. $(x^2 + 8)^2 - 36x^2$
 143. $2t^3 - 16$
 144. $5x^3 + 40$
 145. $4x(2x - 1) + 2(2x - 1)^2$
 146. $5(3 - 4x)^2 - 8(3 - 4x)(5x - 1)$
 147. $2(x + 1)(x - 3)^2 - 3(x + 1)^2(x - 3)$
 148. $7(3x + 2)^2(1 - x)^2 + (3x + 2)(1 - x)^3$
 149. $(2x + 1)^4(2)(3x - 1)(3) + (3x - 1)^2(4)(2x + 1)^3(2)$
 150. $(2x + 5)^3(4)(3x + 2)^3(3) + (3x + 2)^4(3)(2x + 5)^2(2)$
 151. $(x^2 + 5)^4(2)(3x - 1)(3) + (3x - 1)^2(4)(x^2 + 5)^3(2x)$
 152. $(x^2 - 1)^3(2)(4x + 5)(4) + (4x + 5)^2(3)(x^2 - 1)^2(2x)$

153. **Compound Interest** After 2 years, an investment of \$500 compounded annually at an interest rate r will yield an amount of $500(1 + r)^2$.

- (a) Write this polynomial in standard form.
 (b) Use a calculator to evaluate the polynomial for the values of r shown in the table.

r	$2\frac{1}{2}\%$	3%	4%	$4\frac{1}{2}\%$	5%
$500(1 + r)^2$					

- (c) What conclusion can you make from the table?

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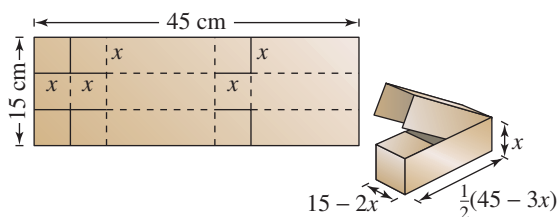
154. Compound Interest After 3 years, an investment of \$1200 compounded annually at an interest rate r will yield an amount of $1200(1 + r)^3$.

- Write this polynomial in standard form.
- Use a calculator to evaluate the polynomial for the values of r shown in the table.

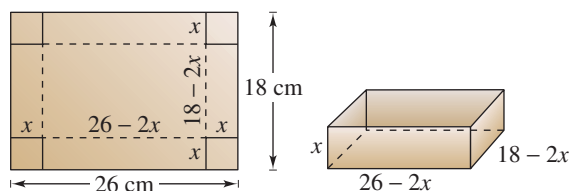
r	2%	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$
$1200(1 + r)^3$					

(c) What conclusion can you make from the table?

155. Geometry An overnight shipping company is designing a closed box by cutting along the solid lines and folding along the broken lines on the rectangular piece of corrugated cardboard shown in the figure. The length and width of the rectangle are 45 centimeters and 15 centimeters, respectively. Find the volume of the box in terms of x . Find the volume when $x = 3$, $x = 5$, and $x = 7$.



156. Geometry A take-out fast food restaurant is constructing an open box made by cutting squares out of the corners of a piece of cardboard that is 18 centimeters by 26 centimeters (see figure). The edge of each cut-out square is x centimeters. Find the volume of the box in terms of x . Find the volume when $x = 1$, $x = 2$, and $x = 3$.



157. Stopping Distance The stopping distance of an automobile is the distance traveled during the driver's reaction time plus the distance traveled after the brakes are applied. In an experiment, these distances were measured (in feet) when the automobile was traveling at a speed of x miles per hour on dry, level pavement, as shown in the bar graph. The distance traveled during the reaction time R was

$$R = 1.1x$$

and the braking distance B was

$$B = 0.0475x^2 - 0.001x + 0.23.$$

- Determine the polynomial that represents the total stopping distance T .
- Use the result of part (a) to estimate the total stopping distance when $x = 30$, $x = 40$, and $x = 55$.
- Use the bar graph to make a statement about the total stopping distance required for increasing speeds.

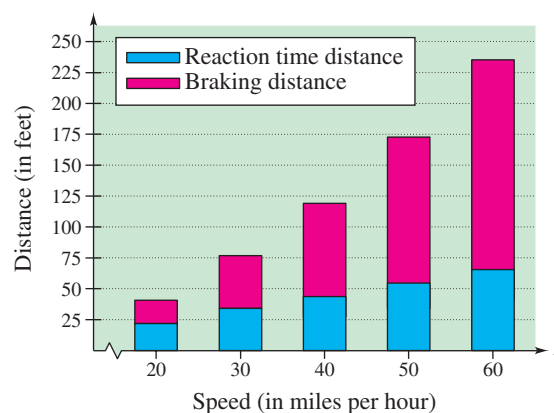


Figure for 157

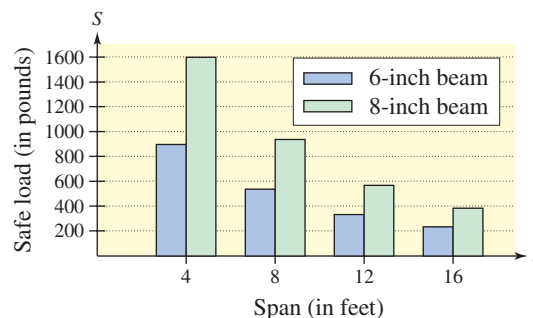
158. Engineering A uniformly distributed load is placed on a one-inch-wide steel beam. When the span of the beam is x feet and its depth is 6 inches, the safe load S (in pounds) is approximated by

$$S_6 = (0.06x^2 - 2.42x + 38.71)^2.$$

When the depth is 8 inches, the safe load is approximated by

$$S_8 = (0.08x^2 - 3.30x + 51.93)^2.$$

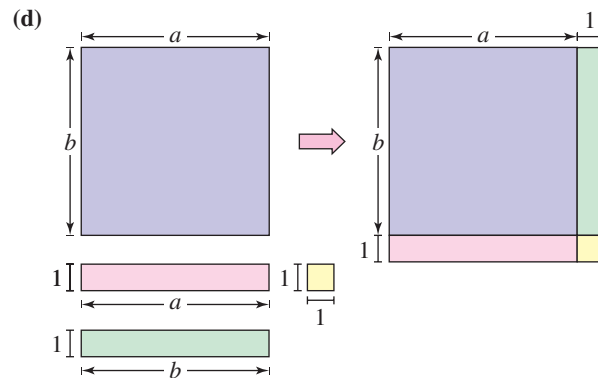
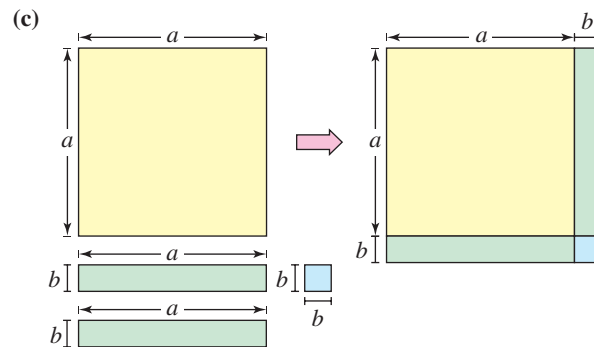
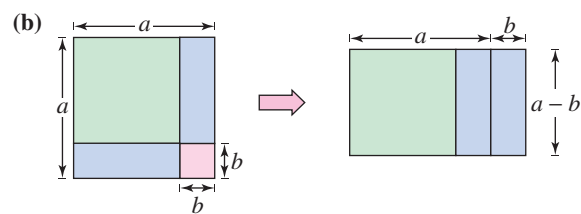
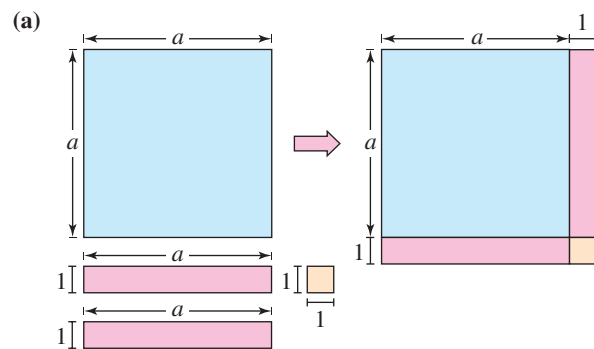
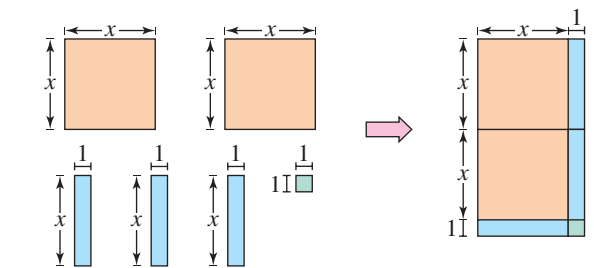
- Use the bar graph to estimate the difference in the safe loads for these two beams when the span is 12 feet.
- How does the difference in safe load change as the span increases?



Geometric Modeling In Exercises 159–162, match the factoring formula with the correct geometric factoring model. [The models are labeled (a), (b), (c), and (d).] For instance, a factoring model for

$$2x^2 + 3x + 1 = (2x + 1)(x + 1)$$

is shown in the figure.

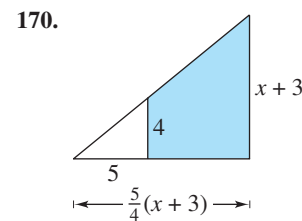
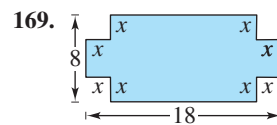
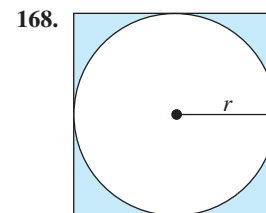
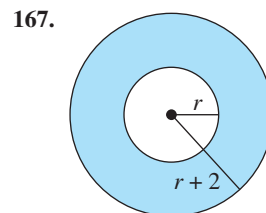


- 159. $a^2 - b^2 = (a + b)(a - b)$
- 160. $a^2 + 2ab + b^2 = (a + b)^2$
- 161. $a^2 + 2a + 1 = (a + 1)^2$
- 162. $ab + a + b + 1 = (a + 1)(b + 1)$

Geometric Modeling In Exercises 163–166, draw a geometric factoring model to represent the factorization.

- 163. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$
- 164. $x^2 + 4x + 3 = (x + 3)(x + 1)$
- 165. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$
- 166. $x^2 + 3x + 2 = (x + 2)(x + 1)$

Geometry In Exercises 167–170, write an expression in factored form for the area of the shaded portion of the figure.



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f In Exercises 171–176, factor the expression completely.

171. $x^4(4)(2x + 1)^3(2x) + (2x + 1)^4(4x^3)$
 172. $x^3(3)(x^2 + 1)^2(2x) + (x^2 + 1)^3(3x^2)$
 173. $(2x - 5)^4(3)(5x - 4)^2(5) + (5x - 4)^3(4)(2x - 5)^3(2)$
 174. $(x^2 - 5)^3(2)(4x + 3)(4) + (4x + 3)^2(3)(x^2 - 5)^2(x^2)$
 175. $\frac{(5x - 1)(3) - (3x + 1)(5)}{(5x - 1)^2}$
 176. $\frac{(2x + 3)(4) - (4x - 1)(2)}{(2x + 3)^2}$

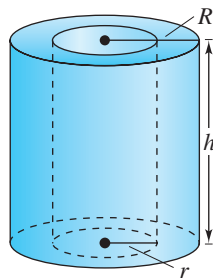
In Exercises 177–180, find all values of b for which the trinomial can be factored with integer coefficients.

177. $x^2 + bx - 15$
 178. $x^2 + bx - 12$
 179. $x^2 + bx + 50$
 180. $x^2 + bx + 24$

In Exercises 181–184, find two integer values of c such that the trinomial can be factored. (There are many correct answers.)

181. $2x^2 + 5x + c$
 182. $3x^2 - x + c$
 183. $3x^2 - 10x + c$
 184. $2x^2 + 9x + c$

185. **Geometry** The cylindrical shell shown in the figure has a volume of $V = \pi R^2 h - \pi r^2 h$.



- (a) Factor the expression for the volume.
 (b) From the result of part (a), show that the volume is

$$2\pi(\text{average radius})(\text{thickness of the shell})h.$$

186. **Chemical Reaction** The rate of change of an autocatalytic chemical reaction is $kQx - kx^2$, where Q is the amount of the original substance, x is the amount of substance formed, and k is a constant of proportionality. Factor the expression.

Synthesis

True or False? In Exercises 187–189, determine whether the statement is true or false. Justify your answer.

187. The product of two binomials is always a second-degree polynomial.
 188. The difference of two perfect squares can be factored as the product of conjugate pairs.
 189. The sum of two perfect squares can be factored as the binomial sum squared.
 190. **Exploration** Find the degree of the product of two polynomials of degrees m and n .
 191. **Exploration** Find the degree of the sum of two polynomials of degrees m and n if $m < n$.
 192. **Writing** Write a paragraph explaining to a classmate why $(x + y)^2 \neq x^2 + y^2$.
 193. **Writing** Write a paragraph explaining to a classmate why $(x - y)^2 \neq x^2 - y^2$.
 194. **Writing** Write a paragraph explaining to a classmate a pattern that can be used to cube a binomial sum. Then use your pattern to cube the sum $(x + y)$.
 195. **Writing** Write a paragraph explaining to a classmate a pattern that can be used to cube a binomial difference. Then use your pattern to cube the difference $(x - y)$.
 196. **Writing** Explain what is meant when it is said that a polynomial is in factored form.
 197. **Think About It** Is $(3x - 6)(x + 1)$ completely factored? Explain.
 198. **Error Analysis** Describe the error.

$$\begin{aligned} 9x^2 - 9x - 54 &= (3x + 6)(3x - 9) \\ &= 3(x + 2)(x - 3) \end{aligned}$$

 199. **Think About It** A third-degree polynomial and a fourth-degree polynomial are added.
 (a) Can the sum be a fourth-degree polynomial? Explain or give an example.
 (b) Can the sum be a second-degree polynomial? Explain or give an example.
 (c) Can the sum be a seventh-degree polynomial? Explain or give an example.
 200. **Think About It** Must the sum of two second-degree polynomials be a second-degree polynomial? If not, give an example.