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1.6 Combinations of Functions

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. If $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, you can form the sum, difference, product, and quotient of f and g as follows.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1)$$
$$= x^2 + 2x - 4 \qquad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 - 1)$$
$$= -x^2 + 2x - 2 \qquad \text{Difference}$$
$$f(x) \cdot g(x) = (2x - 3)(x^2 - 1)$$
$$= 2x^3 - 3x^2 - 2x + 3 \qquad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \qquad \text{Quotient}$$

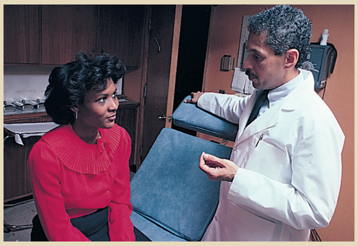
The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

What you should learn

- Add, subtract, multiply, and divide functions.
- Find compositions of one function with another function.
- Use combinations of functions to model and solve real-life problems.

Why you should learn it

Combining functions can sometimes help you better understand the big picture. For instance, Exercises 75 and 76 on page 145 illustrate how to use combinations of functions to analyze U.S. health care expenditures.



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Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

Solution

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x$$

When $x = 2$, the value of this sum is $(f + g)(2) = 2^2 + 4(2) = 12$.

CHECKPOINT Now try Exercise 7(a).

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Example 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Algebraic Solution

The difference of the functions f and g is

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (2x + 1) - (x^2 + 2x - 1)\\ &= -x^2 + 2.\end{aligned}$$

When $x = 2$, the value of this difference is

$$\begin{aligned}(f - g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

Note that $(f - g)(2)$ can also be evaluated as follows.

$$\begin{aligned}(f - g)(2) &= f(2) - g(2)\\ &= [2(2) + 1] - [2^2 + 2(2) - 1]\\ &= 5 - 7\\ &= -2\end{aligned}$$

CHECKPOINT Now try Exercise 7(b).

Graphical Solution

You can use a graphing utility to graph the difference of two functions. Enter the functions as follows (see Figure 1.70).

$$y_1 = 2x + 1$$

$$y_2 = x^2 + 2x - 1$$

$$y_3 = y_1 - y_2$$

Graph y_3 as shown in Figure 1.71. Then use the *value* feature or the *zoom* and *trace* features to estimate that the value of the difference when $x = 2$ is -2.

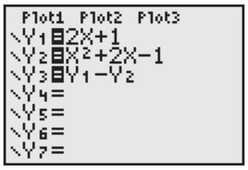

Figure 1.70

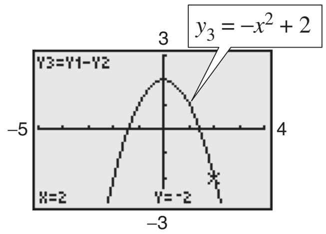

Figure 1.71

In Examples 1 and 2, both f and g have domains that consist of all real numbers. So, the domain of both $(f + g)$ and $(f - g)$ is also the set of all real numbers. Remember that any restrictions on the domains of f or g must be considered when forming the sum, difference, product, or quotient of f and g. For instance, the domain of $f(x) = 1/x$ is all $x \neq 0$, and the domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. This implies that the domain of $(f + g)$ is $(0, \infty)$.

Example 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= (x^2)(x - 3)\\ &= x^3 - 3x^2\end{aligned}$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

CHECKPOINT Now try Exercise 7(c).

Additional Examples

a. Find $(fg)(x)$ given that $f(x) = x + 5$ and $g(x) = 3x$.

Solution

$$\begin{aligned}(fg)(x) &= f(x) \cdot g(x)\\ &= (x + 5)(3x)\\ &= 3x^2 + 15x\end{aligned}$$

b. Find $(gf)(x)$ given that $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x}{x + 1}$.

Solution

$$\begin{aligned}(gf)(x) &= g(x) \cdot f(x)\\ &= \left(\frac{x}{x + 1}\right)\left(\frac{1}{x}\right)\\ &= \frac{1}{x + 1},\ x \neq 0\end{aligned}$$

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Example 4 Finding the Quotient of Two Functions

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions given by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then find the domains of f/g and g/f.

Solution

The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}},$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains for f/g and g/f are as follows.

Domain of (f/g): $[0, 2)$ Domain of (g/f): $(0, 2]$

✓CHECKPOINT Now try Exercise 7(d).

TECHNOLOGY TIP You can confirm the domain of f/g in Example 4 with your graphing utility by entering the three functions $y_1 = \sqrt{x}$, $y_2 = \sqrt{4 - x^2}$, and $y_3 = y_1/y_2$, and graphing y_3, as shown in Figure 1.72. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 2. So, you can estimate the domain of f/g to be $[0, 2)$. You can confirm the domain of g/f in Example 4 by entering $y_4 = y_2/y_1$ and graphing y_4, as shown in Figure 1.73. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 0. So, you can estimate the domain of g/f to be $(0, 2]$.

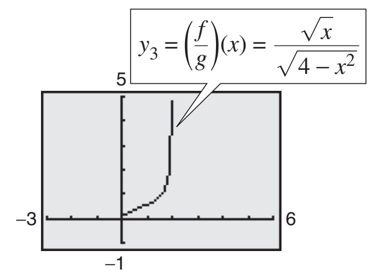

Figure 1.72

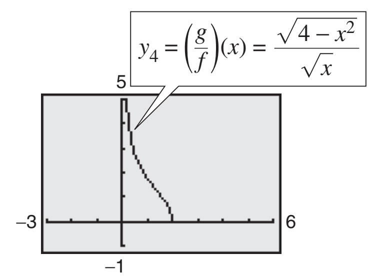

Figure 1.73

Compositions of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, if $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$f(g(x)) = f(x + 1) = (x + 1)^2.$$

This composition is denoted as $f \circ g$ and is read as "f composed with g."

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.74.)

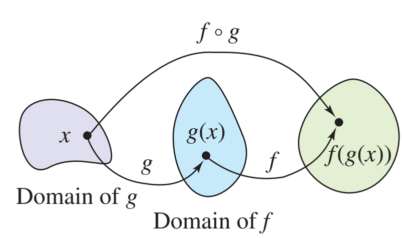

Figure 1.74

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Example 5 Forming the Composition of f with g

Find $(f \circ g)(x)$ for $f(x) = \sqrt{x}$, $x \geq 0$, and $g(x) = x - 1$, $x \geq 1$. If possible, find $(f \circ g)(2)$ and $(f \circ g)(0)$.

Solution

$$(f \circ g)(x) = f(g(x)) \qquad \text{Definition of } f \circ g$$
$$= f(x - 1) \qquad \text{Definition of } g(x)$$
$$= \sqrt{x - 1}, \quad x \geq 1 \qquad \text{Definition of } f(x)$$

The domain of $f \circ g$ is $[1, \infty)$. So, $(f \circ g)(2) = \sqrt{2 - 1} = 1$ is defined, but $(f \circ g)(0)$ is not defined because 0 is not in the domain of $f \circ g$.

CHECKPOINT Now try Exercise 35.

The composition of f with g is generally not the same as the composition of g with f. This is illustrated in Example 6.

Exploration

Let $f(x) = x + 2$ and $g(x) = 4 - x^2$. Are the compositions $f \circ g$ and $g \circ f$ equal? You can use your graphing utility to answer this question by entering and graphing the following functions.

$$y_1 = (4 - x^2) + 2$$
$$y_2 = 4 - (x + 2)^2$$

What do you observe? Which function represents $f \circ g$ and which represents $g \circ f$?

Example 6 Compositions of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, evaluate (a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$ when $x = 0$, 1, 2, and 3.

Algebraic Solution

a. $(f \circ g)(x) = f(g(x))$ Definition of $f \circ g$

$= f(4 - x^2)$ Definition of $g(x)$

$= (4 - x^2) + 2$ Definition of $f(x)$

$= -x^2 + 6$

$(f \circ g)(0) = -0^2 + 6 = 6$

$(f \circ g)(1) = -1^2 + 6 = 5$

$(f \circ g)(2) = -2^2 + 6 = 2$

$(f \circ g)(3) = -3^2 + 6 = -3$

b. $(g \circ f)(x) = g(f(x))$ Definition of $g \circ f$

$= g(x + 2)$ Definition of $f(x)$

$= 4 - (x + 2)^2$ Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$

$= -x^2 - 4x$

$(g \circ f)(0) = -0^2 - 4(0) = 0$

$(g \circ f)(1) = -1^2 - 4(1) = -5$

$(g \circ f)(2) = -2^2 - 4(2) = -12$

$(g \circ f)(3) = -3^2 - 4(3) = -21$

Note that $f \circ g \neq g \circ f$.

CHECKPOINT Now try Exercise 37.

Numerical Solution

a. You can use the *table* feature of a graphing utility to evaluate $f \circ g$ when $x = 0$, 1, 2, and 3. Enter $y_1 = g(x)$ and $y_2 = f(g(x))$ in the *equation editor* (see Figure 1.75). Then set the table to *ask* mode to find the desired function values (see Figure 1.76). Finally, display the table, as shown in Figure 1.77.

b. You can evaluate $g \circ f$ when $x = 0$, 1, 2, and 3 by using a procedure similar to that of part (a). You should obtain the table shown in Figure 1.78.

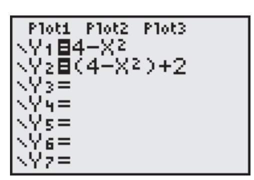

Figure 1.75

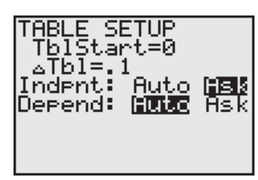

Figure 1.76

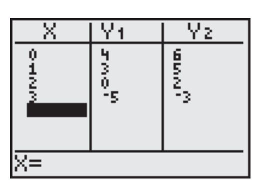

Figure 1.77

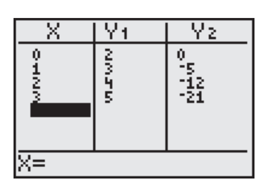

Figure 1.78

From the tables you can see that $f \circ g \neq g \circ f$.

To determine the domain of a composite function $f \circ g$, you need to restrict the outputs of g so that they are in the domain of f. For instance, to find the domain of $f \circ g$ given that $f(x) = 1/x$ and $g(x) = x + 1$, consider the outputs of g. These can be any real number. However, the domain of f is restricted to all real numbers except 0. So, the outputs of g must be restricted to all real numbers except 0. This means that $g(x) \neq 0$, or $x \neq -1$. So, the domain of $f \circ g$ is all real numbers except $x = -1$.

Example 7 Finding the Domain of a Composite Function

Find the domain of the composition $(f \circ g)(x)$ for the functions given by

$$f(x) = x^2 - 9 \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Algebraic Solution

The composition of the functions is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f\left(\sqrt{9 - x^2}\right)\\ &= \left(\sqrt{9 - x^2}\right)^2 - 9\\ &= 9 - x^2 - 9\\ &= -x^2\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $(f \circ g)$ is $[-3, 3]$.

CHECKPOINT Now try Exercise 39.

Graphical Solution

You can use a graphing utility to graph the composition of the functions $(f \circ g)(x)$ as $y = \left(\sqrt{9 - x^2}\right)^2 - 9$. Enter the functions as follows.

$$y_1 = \sqrt{9 - x^2} \qquad y_2 = y_1^2 - 9$$

Graph y_2, as shown in Figure 1.79. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from -3 to 3. So, you can graphically estimate the domain of $(f \circ g)(x)$ to be $[-3, 3]$.

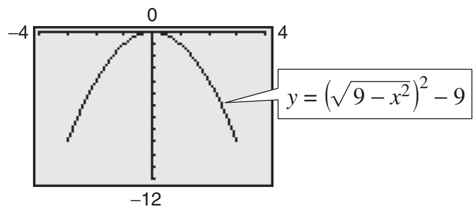

Figure 1.79

Example 8 A Case in Which $f \circ g = g \circ f$

Given $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$, find each composition.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f\left(\frac{1}{2}(x - 3)\right)\\ &= 2\left[\frac{1}{2}(x - 3)\right] + 3\\ &= x - 3 + 3 = x\\ &= x\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(2x + 3)\\ &= \frac{1}{2}\left[(2x + 3) - 3\right]\\ &= \frac{1}{2}(2x)\\ &= x\end{aligned}$$

CHECKPOINT Now try Exercise 51.

STUDY TIP

In Example 8, note that the two composite functions $f \circ g$ and $g \circ f$ are equal, and both represent the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You will study this special case in the next section.

In Examples 5, 6, 7, and 8, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. Basically, to "decompose" a composite function, look for an "inner" and an "outer" function.

Example 9 Identifying a Composite Function

algebra of calculus

Write the function $h(x) = (3x - 5)^3$ as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = 3x - 5$ and the outer function to be $f(x) = x^3$. Then you can write

$$h(x) = (3x - 5)^3$$
$$= f(3x - 5)$$
$$= f(g(x)).$$

CHECKPOINT Now try Exercise 65.

Example 10 Identifying a Composite Function

algebra of calculus

Write the function

$$h(x) = \frac{1}{(x - 2)^2}$$

as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2}$$
$$= x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x - 2)^2}$$
$$= (x - 2)^{-2}$$
$$= f(x - 2)$$
$$= f(g(x)).$$

CHECKPOINT Now try Exercise 69.

Exploration

Write each function as a composition of two functions.

a. $h(x) = |x^3 - 2|$

b. $r(x) = |x^3| - 2$

What do you notice about the inner and outer functions?

Activities

1. Find $(f + g)(-1)$ and $\left(\frac{f}{g}\right)(2)$ for $f(x) = 3x^2 + 2$, $g(x) = 2x$.
   Answer: $3; \frac{7}{2}$
2. Given $f(x) = 3x^2 + 2$ and $g(x) = 2x$, find $f \circ g$.
   Answer: $(f \circ g)(x) = 12x^2 + 2$
3. Find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)
   $h(x) = \frac{1}{\sqrt{3x + 1}}$.
   Answer:
   $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = 3x + 1$

Exploration

The function in Example 10 can be decomposed in other ways. For which of the following pairs of functions is $h(x)$ equal to $f(g(x))$?

a. $g(x) = \frac{1}{x - 2}$ and
$f(x) = x^2$

b. $g(x) = x^2$ and
$f(x) = \frac{1}{x - 2}$

c. $g(x) = \frac{1}{x}$ and
$f(x) = (x - 2)^2$

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Example 11 Bacteria Count

The number N of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 500, \qquad 2 \le T \le 14$$

where T is the temperature of the food (in degrees Celsius). When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \qquad 0 \le t \le 3$$

where t is the time (in hours).

a. Find the composition $N(T(t))$ and interpret its meaning in context.

b. Find the number of bacteria in the food when $t = 2$ hours.

c. Find the time when the bacterial count reaches 2000.

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

The composite function $N(T(t))$ represents the number of bacteria as a function of the amount of time the food has been out of refrigeration.

b. When $t = 2$, the number of bacteria is

$$\begin{aligned} N &= 320(2)^2 + 420 \\ &= 1280 + 420 \\ &= 1700. \end{aligned}$$

c. The bacterial count will reach $N = 2000$ when $320t^2 + 420 = 2000$. You can solve this equation for t algebraically as follows.

$$\begin{aligned} 320t^2 + 420 &= 2000 \\ 320t^2 &= 1580 \\ t^2 &= \frac{79}{16} \\ t &= \frac{\sqrt{79}}{4} \\ t &\approx 2.22 \text{ hours} \end{aligned}$$

So, the count will reach 2000 when $t \approx 2.22$ hours. When you solve this equation, note that the negative value is rejected because it is not in the domain of the composite function. You can use a graphing utility to confirm your solution. First graph the equation $N = 320t^2 + 420$, as shown in Figure 1.80. Then use the *zoom* and *trace* features to approximate $N = 2000$ when $t \approx 2.22$, as shown in Figure 1.81.

CHECKPOINT Now try Exercise 81.

Exploration

Use a graphing utility to graph $y_1 = 320x^2 + 420$ and $y_2 = 2000$ in the same viewing window. (Use a viewing window in which $0 \le x \le 3$ and $400 \le y \le 4000$.) Explain how the graphs can be used to answer the question asked in Example 11(c). Compare your answer with that given in part (c). When will the bacteria count reach 3200?

Notice that the model for this bacteria count situation is valid only for a span of 3 hours. Now suppose that the minimum number of bacteria in the food is reduced from 420 to 100. Will the number of bacteria still reach a level of 2000 within the three-hour time span? Will the number of bacteria reach a level of 3200 within 3 hours?

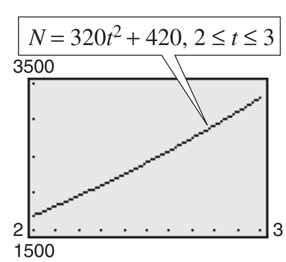

Figure 1.80

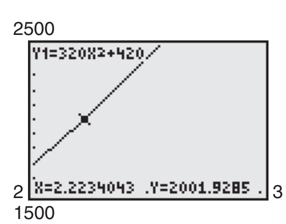

Figure 1.81

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1.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

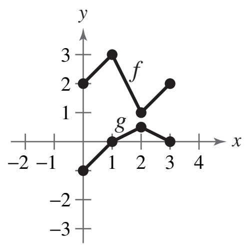
Vocabulary Check

Fill in the blanks.

1. Two functions f and g can be combined by the arithmetic operations of _______ , _______ , _______ , and _______ to create new functions.
2. The _______ of the function f with the function g is $(f \circ g)(x) = f(g(x))$.
3. The domain of $f \circ g$ is the set of all x in the domain of g such that _______ is in the domain of f.
4. To decompose a composite function, look for an _______ and an _______ function.

In Exercises 1–4, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

1.



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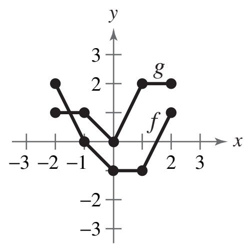

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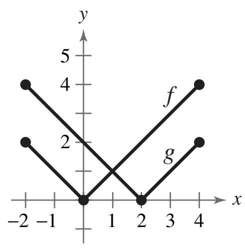

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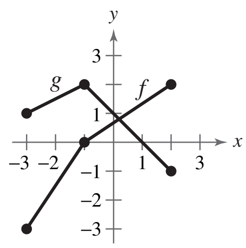

In Exercises 5–12, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

5. $f(x) = x + 3, \quad g(x) = x - 3$
6. $f(x) = 2x - 5, \quad g(x) = 1 - x$
7. $f(x) = x^2, \quad g(x) = 1 - x$
8. $f(x) = 2x - 5, \quad g(x) = 4$
9. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1 - x}$
10. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$
11. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$
12. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

In Exercises 13–26, evaluate the indicated function for $f(x) = x^2 - 1$ and $g(x) = x - 2$ algebraically. If possible, use a graphing utility to verify your answer.

13. $(f + g)(3)$
14. $(f - g)(-2)$
15. $(f - g)(0)$
16. $(f + g)(1)$
17. $(fg)(4)$
18. $(fg)(-6)$
19. $\left(\dfrac{f}{g}\right)(-5)$
20. $\left(\dfrac{f}{g}\right)(0)$
21. $(f - g)(2t)$
22. $(f + g)(t - 4)$
23. $(fg)(-5t)$
24. $(fg)(3t^2)$
25. $\left(\dfrac{f}{g}\right)(-t)$
26. $\left(\dfrac{f}{g}\right)(t + 2)$

In Exercises 27–30, use a graphing utility to graph the functions f, g, and h in the same viewing window.

27. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1, \quad h(x) = f(x) + g(x)$
28. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4, \quad h(x) = f(x) - g(x)$
29. $f(x) = x^2, \quad g(x) = -2x, \quad h(x) = f(x) \cdot g(x)$
30. $f(x) = 4 - x^2, \quad g(x) = x, \quad h(x) = f(x)/g(x)$

In Exercises 31–34, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?

31. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$
32. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$
33. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$
34. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

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In Exercises 35–38, find (a) $f \circ g$, (b) $g \circ f$, and, if possible, (c) $(f \circ g)(0)$.

35. $f(x) = x^2, \quad g(x) = x - 1$

36. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$

37. $f(x) = 3x + 5, \quad g(x) = 5 - x$

38. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

In Exercises 39–48, determine the domains of (a) f, (b) g, and (c) $f \circ g$. Use a graphing utility to verify your results.

39. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

40. $f(x) = \sqrt{x + 3}, \quad g(x) = \dfrac{x}{2}$

41. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$

42. $f(x) = x^{1/4}, \quad g(x) = x^4$

43. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$

44. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{2x}$

45. $f(x) = |x - 4|, \quad g(x) = 3 - x$

46. $f(x) = \dfrac{2}{|x|}, \quad g(x) = x - 1$

47. $f(x) = x + 2, \quad g(x) = \dfrac{1}{x^2 - 4}$

48. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

In Exercises 49–54, (a) find $f \circ g$, $g \circ f$, and the domain of $f \circ g$. (b) Use a graphing utility to graph $f \circ g$ and $g \circ f$. Determine whether $f \circ g = g \circ f$.

49. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

50. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3 - 1$

51. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 9$

52. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$

53. $f(x) = x^{2/3}, \quad g(x) = x^6$

54. $f(x) = |x|, \quad g(x) = -x^2 + 1$

In Exercises 55–60, (a) find $(f \circ g)(x)$ and $(g \circ f)(x)$, (b) determine algebraically whether $(f \circ g)(x) = (g \circ f)(x)$, and (c) use a graphing utility to complete a table of values for the two compositions to confirm your answers to part (b).

55. $f(x) = 5x + 4, \quad g(x) = 4 - x$

56. $f(x) = \frac{1}{4}(x - 1), \quad g(x) = 4x + 1$

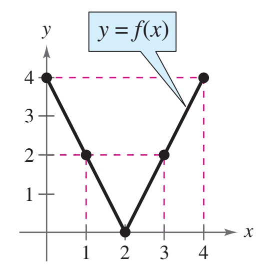
57. $f(x) = \sqrt{x + 6}, \quad g(x) = x^2 - 5$

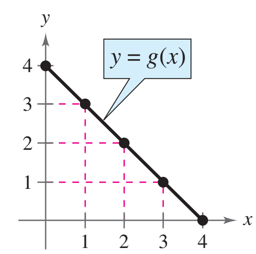
58. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 10}$

59. $f(x) = |x|, \quad g(x) = 2x^3$

60. $f(x) = \dfrac{6}{3x - 5}, \quad g(x) = -x$

In Exercises 61–64, use the graphs of f and g to evaluate the functions.





61. (a) $(f + g)(3)$ (b) $(f/g)(2)$

62. (a) $(f - g)(1)$ (b) $(fg)(4)$

63. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

64. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

In Exercises 65–72, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

65. $h(x) = (2x + 1)^2$

66. $h(x) = (1 - x)^3$

67. $h(x) = \sqrt[3]{x^2 - 4}$

68. $h(x) = \sqrt{9 - x}$

69. $h(x) = \dfrac{1}{x + 2}$

70. $h(x) = \dfrac{4}{(5x + 2)^2}$

71. $h(x) = (x + 4)^2 + 2(x + 4)$

72. $h(x) = (x + 3)^{3/2} + 4(x + 3)^{1/2}$

73. ***Stopping Distance*** The research and development department of an automobile manufacturer has determined that when required to stop quickly to avoid an accident, the distance (in feet) a car travels during the driver's reaction time is given by

$$R(x) = \frac{3}{4}x$$

where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is given by

$$B(x) = \frac{1}{15}x^2.$$

(a) Find the function that represents the total stopping distance T.

(b) Use a graphing utility to graph the functions R, B, and T in the same viewing window for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

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74. ***Sales*** From 2000 to 2006, the sales R_1 (in thousands of dollars) for one of two restaurants owned by the same parent company can be modeled by $R_1 = 480 - 8t - 0.8t^2$, for $t = 0, 1, 2, 3, 4, 5, 6$, where $t = 0$ represents 2000. During the same seven-year period, the sales R_2 (in thousands of dollars) for the second restaurant can be modeled by $R_2 = 254 + 0.78t$, for $t = 0, 1, 2, 3, 4, 5, 6$.

(a) Write a function R_3 that represents the total sales for the two restaurants.

(b) Use a graphing utility to graph R_1, R_2, and R_3 (the total sales function) in the same viewing window.

Data Analysis **In Exercises 75 and 76, use the table, which shows the total amounts spent (in billions of dollars) on health services and supplies in the United States and Puerto Rico for the years 1995 through 2005. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively.** (Source: U.S. Centers for Medicare and Medicaid Services)

Year	y_1	y_2	y_3
1995	146	330	457
1996	152	344	483
1997	162	361	503
1998	176	385	520
1999	185	414	550
2000	193	451	592
2001	202	497	655
2002	214	550	718
2003	231	601	766
2004	246	647	824
2005	262	691	891

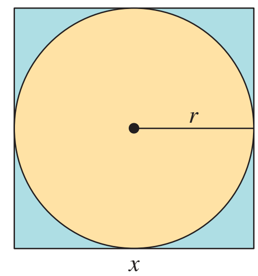
The models for this data are $y_1 = 11.4t + 83$, $y_2 = 2.31t^2 - 8.4t + 310$, and $y_3 = 3.03t^2 - 16.8t + 467$, where t represents the year, with $t = 5$ corresponding to 1995.

75. Use the models and the *table* feature of a graphing utility to create a table showing the values of y_1, y_2, and y_3 for each year from 1995 to 2005. Compare these values with the original data. Are the models a good fit? Explain.

76. Use a graphing utility to graph y_1, y_2, y_3, and $y_T = y_1 + y_2 + y_3$ in the same viewing window. What does the function y_T represent? Explain.

77. ***Ripples*** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outermost ripple is given by $r(t) = 0.6t$, where t is the time (in seconds) after the pebble strikes the water. The area of the circle is given by $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

78. ***Geometry*** A square concrete foundation was prepared as a base for a large cylindrical gasoline tank (see figure).



(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

79. ***Cost*** The weekly cost C of producing x units in a manufacturing process is given by

$$C(x) = 60x + 750.$$

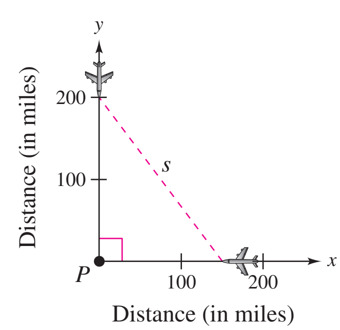
The number of units x produced in t hours is $x(t) = 50t$.

(a) Find and interpret $C(x(t))$.

(b) Find the number of units produced in 4 hours.

(c) Use a graphing utility to graph the cost as a function of time. Use the *trace* feature to estimate (to two-decimal-place accuracy) the time that must elapse until the cost increases to \$15,000.

80. ***Air Traffic Control*** An air traffic controller spots two planes at the same altitude flying toward each other. Their flight paths form a right angle at point P. One plane is 150 miles from point P and is moving at 450 miles per hour. The other plane is 200 miles from point P and is moving at 450 miles per hour. Write the distance s between the planes as a function of time t.



81. ***Bacteria*** The number of bacteria in a refrigerated food product is given by $N(T) = 10T^2 - 20T + 600$, for $1 \le T \le 20$, where T is the temperature of the food in degrees Celsius. When the food is removed from the refrigerator, the temperature of the food is given by $T(t) = 2t + 1$, where t is the time in hours.

(a) Find the composite function $N(T(t))$ or $(N \circ T)(t)$ and interpret its meaning in the context of the situation.

(b) Find $(N \circ T)(6)$ and interpret its meaning.

(c) Find the time when the bacteria count reaches 800.

82. ***Pollution*** The spread of a contaminant is increasing in a circular pattern on the surface of a lake. The radius of the contaminant can be modeled by $r(t) = 5.25\sqrt{t}$, where r is the radius in meters and t is time in hours since contamination.

(a) Find a function that gives the area A of the circular leak in terms of the time t since the spread began.

(b) Find the size of the contaminated area after 36 hours.

(c) Find when the size of the contaminated area is 6250 square meters.

83. ***Salary*** You are a sales representative for an automobile manufacturer. You are paid an annual salary plus a bonus of 3% of your sales over \$500,000. Consider the two functions

$f(x) = x - 500{,}000$ and $g(x) = 0.03x$.

If x is greater than \$500,000, which of the following represents your bonus? Explain.

(a) $f(g(x))$ (b) $g(f(x))$

84. ***Consumer Awareness*** The suggested retail price of a new car is p dollars. The dealership advertised a factory rebate of \$1200 and an 8% discount.

(a) Write a function R in terms of p giving the cost of the car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(18{,}400)$ and $(S \circ R)(18{,}400)$. Which yields the lower cost for the car? Explain.

Synthesis

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. If $f(x) = x + 1$ and $g(x) = 6x$, then

$(f \circ g)(x) = (g \circ f)(x)$.

86. If you are given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

Exploration **In Exercises 87 and 88, three siblings are of three different ages. The oldest is twice the age of the middle sibling, and the middle sibling is six years older than one-half the age of the youngest.**

87. (a) Write a composite function that gives the oldest sibling's age in terms of the youngest. Explain how you arrived at your answer.

(b) If the oldest sibling is 16 years old, find the ages of the other two siblings.

88. (a) Write a composite function that gives the youngest sibling's age in terms of the oldest. Explain how you arrived at your answer.

(b) If the youngest sibling is two years old, find the ages of the other two siblings.

89. ***Proof*** Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

90. ***Conjecture*** Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

91. ***Proof*** Given a function f, prove that $g(x)$ is even and $h(x)$ is odd, where $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and $h(x) = \frac{1}{2}[f(x) - f(-x)]$.

92. (a) Use the result of Exercise 91 to prove that any function can be written as a sum of even and odd functions. (*Hint:* Add the two equations in Exercise 91.)

(b) Use the result of part (a) to write each function as a sum of even and odd functions.

$f(x) = x^2 - 2x + 1, \quad g(x) = \dfrac{1}{x + 1}$

Skills Review

In Exercises 93–96, find three points that lie on the graph of the equation. (There are many correct answers.)

93. $y = -x^2 + x - 5$

94. $y = \frac{1}{5}x^3 - 4x^2 + 1$

95. $x^2 + y^2 = 24$

96. $y = \dfrac{x}{x^2 - 5}$

In Exercises 97–100, find an equation of the line that passes through the two points.

97. $(-4, -2), (-3, 8)$

98. $(1, 5), (-8, 2)$

99. $\left(\frac{3}{2}, -1\right), \left(-\frac{1}{3}, 4\right)$

100. $(0, 1.1), (-4, 3.1)$