

## 2.3 Complex Numbers

### The Imaginary Unit $i$

Some quadratic equations have no real solutions. For instance, the quadratic equation  $x^2 + 1 = 0$  has no real solution because there is no real number  $x$  that can be squared to produce  $-1$ . To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit  $i$** , defined as

$$i = \sqrt{-1} \quad \text{Imaginary unit}$$

where  $i^2 = -1$ . By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers**. Each complex number can be written in the **standard form**  $a + bi$ . For instance, the standard form of the complex number  $\sqrt{-9} - 5$  is  $-5 + 3i$  because

$$\sqrt{-9} - 5 = \sqrt{3^2(-1)} - 5 = 3\sqrt{-1} - 5 = 3i - 5 = -5 + 3i.$$

In the standard form  $a + bi$ , the real number  $a$  is called the **real part** of the **complex number**  $a + bi$ , and the number  $bi$  (where  $b$  is a real number) is called the **imaginary part** of the complex number.

#### Definition of a Complex Number

If  $a$  and  $b$  are real numbers, the number  $a + bi$  is a **complex number**, and it is said to be written in **standard form**. If  $b = 0$ , the number  $a + bi = a$  is a real number. If  $b \neq 0$ , the number  $a + bi$  is called an **imaginary number**. A number of the form  $bi$ , where  $b \neq 0$ , is called a **pure imaginary number**.

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.20. This is true because every real number  $a$  can be written as a complex number using  $b = 0$ . That is, for every real number  $a$ , you can write  $a = a + 0i$ .

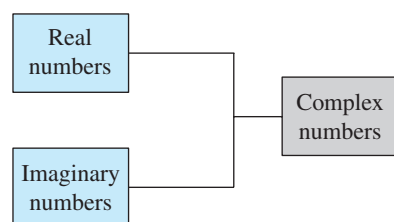


Figure 2.20

#### Equality of Complex Numbers

Two complex numbers  $a + bi$  and  $c + di$ , written in standard form, are equal to each other

$$a + bi = c + di \quad \text{Equality of two complex numbers}$$

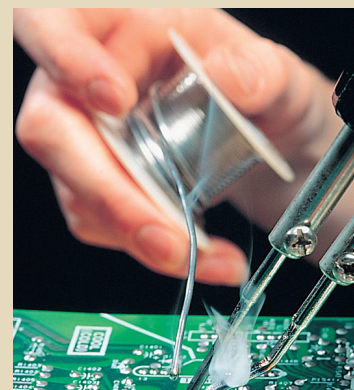
if and only if  $a = c$  and  $b = d$ .

#### What you should learn

- Use the imaginary unit  $i$  to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Plot complex numbers in the complex plane.

#### Why you should learn it

Complex numbers are used to model numerous aspects of the natural world, such as the impedance of an electrical circuit, as shown in Exercises 79 and 80 on page 194.



Phil Degginger/Getty Images

## Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

### Addition and Subtraction of Complex Numbers

If  $a + bi$  and  $c + di$  are two complex numbers written in standard form, their sum and difference are defined as follows.

$$\text{Sum: } (a + bi) + (c + di) = (a + c) + (b + d)i$$

$$\text{Difference: } (a + bi) - (c + di) = (a - c) + (b - d)i$$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number  $a + bi$  is

$$-(a + bi) = -a - bi. \quad \text{Additive inverse}$$

So, you have  $(a + bi) + (-a - bi) = 0 + 0i = 0$ .

### Example 1 Adding and Subtracting Complex Numbers

- a.  $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$  Remove parentheses.  
 $= 3 + 2 - i + 3i$  Group like terms.  
 $= (3 + 2) + (-1 + 3)i$   
 $= 5 + 2i$  Write in standard form.
- b.  $\sqrt{-4} + (-4 - \sqrt{-4}) = 2i + (-4 - 2i)$  Write in  $i$ -form.  
 $= 2i - 4 - 2i$  Remove parentheses.  
 $= -4 + 2i - 2i$  Group like terms.  
 $= -4$  Write in standard form.
- c.  $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$   
 $= 3 + 2 - 5 - 3i + i$   
 $= 0 - 2i$   
 $= -2i$
- d.  $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$   
 $= 3 + 4 - 7 + 2i - i - i$   
 $= 0 + 0i$   
 $= 0$

 **CHECKPOINT** Now try Exercise 19.

### Prerequisite Skills

Review operations on polynomials in Section P.3, if you have difficulty with this example.

For each operation on complex numbers, you can show the parallel operations on polynomials.

### STUDY TIP

In Examples 1(b) and 1(d), note that the sum of complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication  
Commutative Properties of Addition and Multiplication  
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property} \\ &= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property} \\ &= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1 \\ &= ac - bd + (ad)i + (bc)i && \text{Commutative Property} \\ &= (ac - bd) + (ad + bc)i && \text{Associative Property}\end{aligned}$$

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method discussed in Section P.3.

### Example 2 Multiplying Complex Numbers

- a.**  $\sqrt{-4} \cdot \sqrt{-16} = (2i)(4i)$  Write each factor in  $i$ -form.
- $$\begin{aligned}&= 8i^2 && \text{Multiply.} \\ &= 8(-1) && i^2 = -1 \\ &= -8 && \text{Simplify.}\end{aligned}$$
- b.**  $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2$  Product of binomials
- $$\begin{aligned}&= 8 + 6i - 4i - 3(-1) && i^2 = -1 \\ &= 8 + 3 + 6i - 4i && \text{Group like terms.} \\ &= 11 + 2i && \text{Write in standard form.}\end{aligned}$$
- c.**  $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$  Product of binomials
- $$\begin{aligned}&= 9 - 4(-1) && i^2 = -1 \\ &= 9 + 4 && \text{Simplify.} \\ &= 13 && \text{Write in standard form.}\end{aligned}$$
- d.**  $4i(-1 + 5i) = 4i(-1) + 4i(5i)$  Distributive Property
- $$\begin{aligned}&= -4i + 20i^2 && \text{Simplify.} \\ &= -4i + 20(-1) && i^2 = -1 \\ &= -20 - 4i && \text{Write in standard form.}\end{aligned}$$
- e.**  $(3 + 2i)^2 = 9 + 6i + 6i + 4i^2$  Product of binomials
- $$\begin{aligned}&= 9 + 12i + 4(-1) && i^2 = -1 \\ &= 9 - 4 + 12i && \text{Group like terms.} \\ &= 5 + 12i && \text{Write in standard form.}\end{aligned}$$

 **CHECKPOINT** Now try Exercise 29.

### Exploration

Complete the following:

$$\begin{array}{ll}i^1 = i & i^7 = \square \\ i^2 = -1 & i^8 = \square \\ i^3 = -i & i^9 = \square \\ i^4 = 1 & i^{10} = \square \\ i^5 = \square & i^{11} = \square \\ i^6 = \square & i^{12} = \square\end{array}$$

What pattern do you see?  
Write a brief description of how you would find  $i$  raised to any positive integer power.

### STUDY TIP

Before you perform operations with complex numbers, be sure to rewrite the terms or factors in  $i$ -form first and then proceed with the operations, as shown in Example 2(a).

## Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the forms  $a + bi$  and  $a - bi$ , called **complex conjugates**.

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2$$

### Example 3 Multiplying Conjugates

Multiply  $3 - 5i$  by its complex conjugate.

#### Solution

The complex conjugate of  $3 - 5i$  is  $3 + 5i$ .

$$(3 - 5i)(3 + 5i) = 3^2 - (5i)^2 = 9 - 25i^2 = 9 - 25(-1) = 34$$

 **CHECKPOINT** Now try Exercise 37.

To write the quotient of  $a + bi$  and  $c + di$  in standard form, where  $c$  and  $d$  are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \left( \frac{c - di}{c - di} \right) \\ &= \frac{ac + bd}{c^2 + d^2} + \left( \frac{bc - ad}{c^2 + d^2} \right) i. \end{aligned}$$

Multiply numerator and denominator by complex conjugate of denominator.

Standard form

A comparison with the method of rationalizing denominators (Section P.2) may be helpful.

### Example 4 Writing a Quotient of Complex Numbers in Standard Form

Write the quotient  $\frac{2 + 3i}{4 - 2i}$  in standard form.

#### Solution

$$\begin{aligned} \frac{2 + 3i}{4 - 2i} &= \frac{2 + 3i}{4 - 2i} \left( \frac{4 + 2i}{4 + 2i} \right) \\ &= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} \\ &= \frac{8 - 6 + 16i}{16 + 4} \\ &= \frac{2 + 16i}{20} \\ &= \frac{1}{10} + \frac{4}{5}i \end{aligned}$$

Multiply numerator and denominator by complex conjugate of denominator.

Expand.

$i^2 = -1$

Simplify.

Write in standard form.

 **CHECKPOINT** Now try Exercise 49.

#### TECHNOLOGY TIP

Some graphing utilities can perform operations with complex numbers. For instance, on some graphing utilities, to divide  $2 + 3i$  by  $4 - 2i$ , use the following keystrokes.

$\boxed{2} \boxed{+} \boxed{3} \boxed{i} \boxed{\div}$

$\boxed{4} \boxed{-} \boxed{2} \boxed{i} \boxed{\div} \boxed{\text{ENTER}}$

The display will be as follows.

$.1 + .8i$  or  $\frac{1}{10} + \frac{4}{5}i$

## Fractals and the Mandelbrot Set

Most applications involving complex numbers are either theoretical or very technical, and are therefore not appropriate for inclusion in this text. However, to give you some idea of how complex numbers can be used in applications, a general description of their use in **fractal geometry** is presented.

To begin, consider a coordinate system called the **complex plane**. Just as every real number corresponds to a point on the real number line, every complex number corresponds to a point in the complex plane, as shown in Figure 2.21. In this figure, note that the vertical axis is called the **imaginary axis** and the horizontal axis is called the **real axis**. The point that corresponds to the complex number  $a + bi$  is  $(a, b)$ .

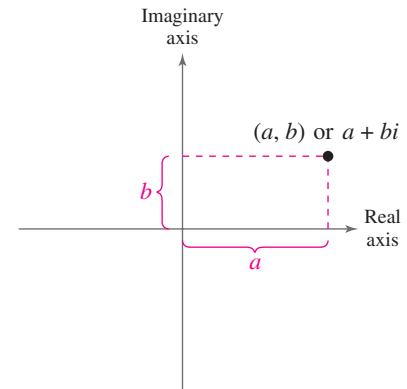


Figure 2.21

### Example 5 Plotting Complex Numbers

Plot each complex number in the complex plane.

- a.  $2 + 3i$       b.  $-1 + 2i$       c.  $4$       d.  $-3i$

#### Solution

- a. To plot the complex number  $2 + 3i$ , move (from the origin) two units to the right on the real axis and then three units upward, as shown in Figure 2.22. In other words, plotting the complex number  $2 + 3i$  in the complex plane is comparable to plotting the point  $(2, 3)$  in the Cartesian plane. (Note that in Figure 2.22,  $i$  is called the imaginary unit because it is located one unit from the origin on the imaginary axis of the complex plane.)
- b. The complex number  $-1 + 2i$  corresponds to the point  $(-1, 2)$ , as shown in Figure 2.22.
- c. The complex number  $4$  corresponds to the point  $(4, 0)$ , as shown in Figure 2.22.
- d. The complex number  $-3i$  corresponds to the point  $(0, -3)$ , as shown in Figure 2.22.

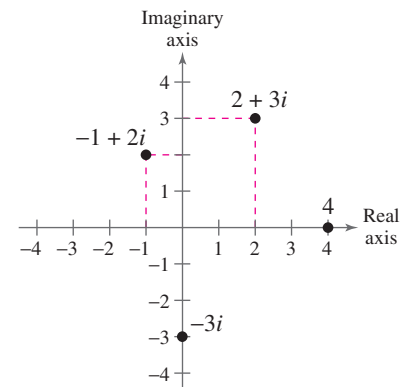


Figure 2.22

**CHECKPOINT** Now try Exercise 71.

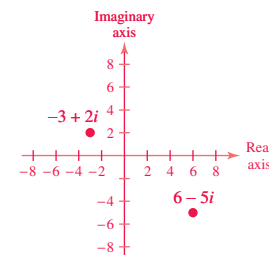
In the hands of a person who understands fractal geometry, the complex plane can become an easel on which stunning pictures, called **fractals**, can be drawn. The most famous such picture is called the **Mandelbrot Set**, named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

$$c, c^2 + c, (c^2 + c)^2 + c, [(c^2 + c)^2 + c]^2 + c, \dots$$

The behavior of this sequence depends on the value of the complex number  $c$ . For some values of  $c$  this sequence is **bounded**, which means that the absolute value of each number ( $|a + bi| = \sqrt{a^2 + b^2}$ ) in the sequence is less than some fixed number  $N$ . For other values of  $c$ , the sequence is **unbounded**, which means that the absolute values of the terms of the sequence become infinitely large. If the sequence is bounded, the complex number  $c$  is in the Mandelbrot Set; if the sequence is unbounded, the complex number  $c$  is not in the Mandelbrot Set.

#### Activities

- Find the product and write the result in standard form.  
 $(4 - \sqrt{-9})(2 + \sqrt{-9})$   
 Answer:  $17 + 6i$
- Write  $\frac{3 + i}{i}$  in standard form.  
 Answer:  $1 - 3i$
- Plot  $6 - 5i$  and  $-3 + 2i$  in the complex plane.



**Example 6** Members of the Mandelbrot Set

- a. The complex number  $-2$  is in the Mandelbrot Set, because for  $c = -2$ , the corresponding Mandelbrot sequence is  $-2, 2, 2, 2, 2, 2, \dots$ , which is bounded.
- b. The complex number  $i$  is also in the Mandelbrot Set, because for  $c = i$ , the corresponding Mandelbrot sequence is

$$i, \quad -1 + i, \quad -i, \quad -1 + i, \quad -i, \quad -1 + i, \quad \dots$$

which is bounded.

- c. The complex number  $1 + i$  is not in the Mandelbrot Set, because for  $c = 1 + i$ , the corresponding Mandelbrot sequence is

$$1 + i, \quad 1 + 3i, \quad -7 + 7i, \quad 1 - 97i, \quad -9407 - 193i, \\ 88,454,401 + 3,631,103i, \quad \dots$$

which is unbounded.

 **CHECKPOINT** Now try Exercise 77.

With this definition, a picture of the Mandelbrot Set would have only two colors: one color for points that are in the set (the sequence is bounded), and one color for points that are outside the set (the sequence is unbounded). Figure 2.23 shows a black and yellow picture of the Mandelbrot Set. The points that are black are in the Mandelbrot Set and the points that are yellow are not.

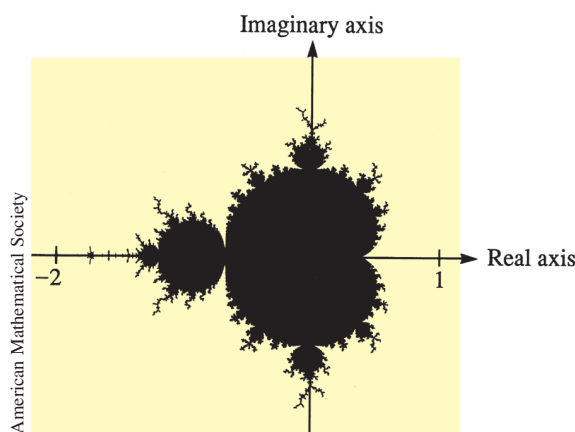


Figure 2.23

To add more interest to the picture, computer scientists discovered that the points that are not in the Mandelbrot Set can be assigned a variety of colors, depending on “how quickly” their sequences diverge (become infinitely large). Figure 2.24 shows three different appendages of the Mandelbrot Set. (The black portions of the picture represent points that are in the Mandelbrot Set.)

Figure 2.25 shows another type of fractal. From this picture, you can see why fractals have fascinated people since their discovery (around 1980). The program for creating the fractal fern on a graphing utility is available at this textbook’s *Online Study Center*.

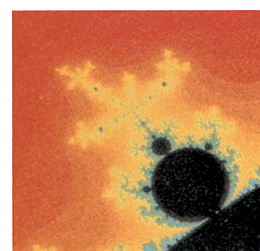
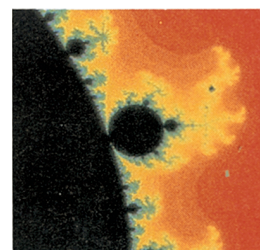
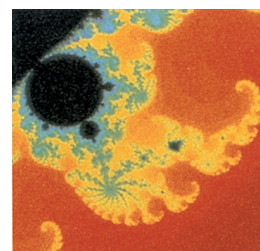


Figure 2.24



Figure 2.25 A Fractal Fern

## 2.3 Exercises

See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

### Vocabulary Check

- Match the type of complex number with its definition.
 

(a) real number	(i) $a + bi$ , $a = 0$ , $b \neq 0$
(b) imaginary number	(ii) $a + bi$ , $b = 0$
(c) pure imaginary number	(iii) $a + bi$ , $a \neq 0$ , $b \neq 0$

In Exercises 2–5, fill in the blanks.

- The imaginary unit  $i$  is defined as  $i = \underline{\hspace{2cm}}$ , where  $i^2 = \underline{\hspace{2cm}}$ .
- The set of real multiples of the imaginary unit  $i$  combined with the set of real numbers is called the set of  $\underline{\hspace{2cm}}$  numbers, which are written in the standard form  $\underline{\hspace{2cm}}$ .
- Complex numbers can be plotted in the complex plane, where the horizontal axis is the  $\underline{\hspace{2cm}}$  axis and the vertical axis is the  $\underline{\hspace{2cm}}$  axis.
- The most famous fractal is called the  $\underline{\hspace{2cm}}$ .

In Exercises 1–4, find real numbers  $a$  and  $b$  such that the equation is true.

- $a + bi = -9 + 4i$
- $a + bi = 12 + 5i$
- $(a - 1) + (b + 3)i = 5 + 8i$
- $(a + 6) + 2bi = 6 - 5i$

In Exercises 5–14, write the complex number in standard form.

- $5 + \sqrt{-16}$
- $2 - \sqrt{-9}$
- $-6$
- $8$
- $-5i + i^2$
- $-3i^2 + i$
- $(\sqrt{-75})^2$
- $(\sqrt{-4})^2 - 7$
- $\sqrt{-0.09}$
- $\sqrt{-0.0004}$

In Exercises 15–24, perform the addition or subtraction and write the result in standard form.

- $(4 + i) - (7 - 2i)$
- $(11 - 2i) - (-3 + 6i)$
- $(-1 + \sqrt{-8}) + (8 - \sqrt{-50})$
- $(7 + \sqrt{-18}) + (3 + \sqrt{-32})$
- $13i - (14 - 7i)$
- $22 + (-5 + 8i) - 10i$
- $(\frac{3}{2} + \frac{5}{2}i) + (\frac{5}{3} + \frac{11}{3}i)$
- $(\frac{3}{4} + \frac{7}{5}i) - (\frac{5}{6} - \frac{1}{6}i)$
- $(1.6 + 3.2i) + (-5.8 + 4.3i)$
- $-(-3.7 - 12.8i) - (6.1 - \sqrt{-24.5})$

In Exercises 25–36, perform the operation and write the result in standard form.

- $\sqrt{-6} \cdot \sqrt{-2}$
- $\sqrt{-5} \cdot \sqrt{-10}$
- $(\sqrt{-10})^2$
- $(\sqrt{-75})^2$

- $(1 + i)(3 - 2i)$
- $(6 - 2i)(2 - 3i)$
- $4i(8 + 5i)$
- $-3i(6 - i)$
- $(\sqrt{14} + \sqrt{10}i)(\sqrt{14} - \sqrt{10}i)$
- $(3 + \sqrt{-5})(7 - \sqrt{-10})$
- $(4 + 5i)^2 - (4 - 5i)^2$
- $(1 - 2i)^2 - (1 + 2i)^2$

In Exercises 37–44, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

- $4 + 3i$
- $7 - 5i$
- $-6 - \sqrt{5}i$
- $-3 + \sqrt{2}i$
- $\sqrt{-20}$
- $\sqrt{-13}$
- $3 - \sqrt{-2}$
- $1 + \sqrt{-8}$

In Exercises 45–52, write the quotient in standard form.

- $\frac{6}{i}$
- $-\frac{5}{2i}$
- $\frac{2}{4 - 5i}$
- $\frac{3}{1 - i}$
- $\frac{2 + i}{2 - i}$
- $\frac{8 - 7i}{1 - 2i}$
- $\frac{i}{(4 - 5i)^2}$
- $\frac{5i}{(2 + 3i)^2}$

In Exercises 53–56, perform the operation and write the result in standard form.

- $\frac{2}{1 + i} - \frac{3}{1 - i}$
- $\frac{2i}{2 + i} + \frac{5}{2 - i}$

## 194 Chapter 2 Solving Equations and Inequalities

55.  $\frac{i}{3-2i} + \frac{2i}{3+8i}$

56.  $\frac{1+i}{i} - \frac{3}{4-i}$

In Exercises 57–62, simplify the complex number and write it in standard form.

57.  $-6i^3 + i^2$

58.  $4i^2 - 2i^3$

59.  $(\sqrt{-75})^3$

60.  $(\sqrt{-2})^6$

61.  $\frac{1}{i^3}$

62.  $\frac{1}{(2i)^3}$

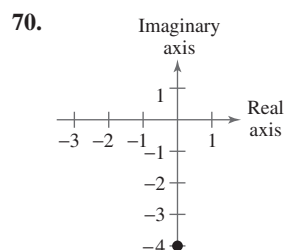
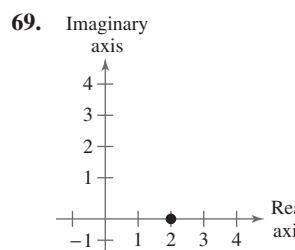
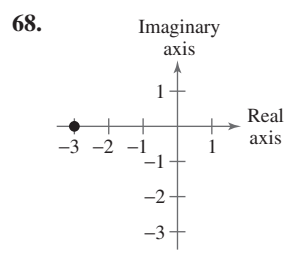
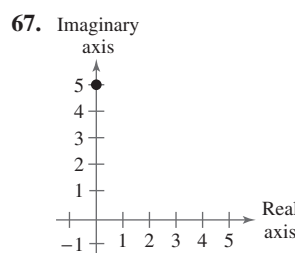
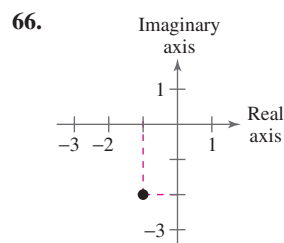
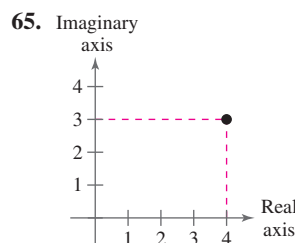
63. Cube each complex number. What do you notice?

(a) 2 (b)  $-1 + \sqrt{3}i$  (c)  $-1 - \sqrt{3}i$

64. Raise each complex number to the fourth power and simplify.

(a) 2 (b)  $-2$  (c)  $2i$  (d)  $-2i$

In Exercises 65–70, determine the complex number shown in the complex plane.



In Exercises 71–76, plot the complex number in the complex plane.

71.  $4 - 5i$

72.  $-7 + 2i$

73.  $3i$

74.  $-5i$

75. 1

76.  $-6$

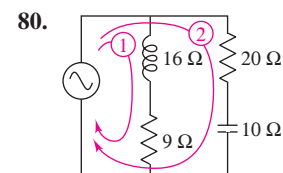
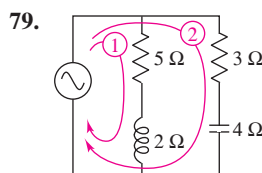
**Fractals** In Exercises 77 and 78, find the first six terms of the sequence given on page 191. From the terms, do you think the given complex number is in the Mandelbrot Set? Explain your reasoning.

77.  $c = \frac{1}{2}i$

78.  $c = 2$

**Impedance** In Exercises 79 and 80, use the following information. The opposition to current in an electrical circuit is called its impedance. The impedance  $z$  in a parallel circuit with two pathways satisfies the equation  $1/z = 1/z_1 + 1/z_2$  where  $z_1$  is the impedance (in ohms) of pathway 1 and  $z_2$  is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of the parallel circuit. (*Hint:* You can find the impedance of each pathway in a parallel circuit by adding the impedances of all components in the pathway.)

	Resistor	Inductor	Capacitor
Symbol			
	$a \Omega$	$b \Omega$	$c \Omega$
Impedance	$a$	$bi$	$-ci$



### Synthesis

**True or False?** In Exercises 81–86, determine whether the statement is true or false. Justify your answer.

81. There is no complex number that is equal to its conjugate.

82.  $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

83. The sum of two imaginary numbers is always an imaginary number.

84. The product of two imaginary numbers is always an imaginary number.

85. The conjugate of the product of two complex numbers is equal to the product of the conjugates of the two complex numbers.

86. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.

### Skills Review

In Exercises 87–90, perform the operation and write the result in standard form.

87.  $(4x - 5)(4x + 5)$

88.  $(x + 2)^3$

89.  $(3x - \frac{1}{2})(x + 4)$

90.  $(2x - 5)^2$