3.4 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n zeros*. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem (See the proof on page 332.)

If $f(x)$ is a polynomial of degree n, where $n > 0$, f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, . . . , c_n$ are complex numbers.

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques.

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Examples 1 and 2 illustrate several cases.

What you should learn

- Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function.
- Find all zeros of polynomial functions, including complex zeros.
- Find conjugate pairs of complex zeros.
- Find zeros of polynomials by factoring.

Why you should learn it

Being able to find zeros of polynomial functions is an important part of modeling real-life problems. For instance, Exercise 63 on page 297 shows how to determine whether a ball thrown with a given velocity can reach a certain height.

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Example 1 Real Zeros of a Polynomial Function

Counting multiplicity, confirm that the second-degree polynomial function

$$f(x) = x^2 - 6x + 9$$

has exactly *two* zeros: $x = 3$ and $x = 3$.

Solution

$$x^2 - 6x + 9 = (x - 3)^2 = 0$$

$$x - 3 = 0 \quad \Rightarrow \quad x = 3 \qquad \text{Repeated solution}$$

The graph in Figure 3.38 touches the x-axis at $x = 3$.

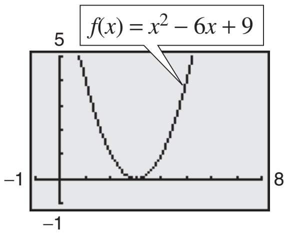


Figure 3.38

CHECKPOINT Now try Exercise 1.

Example 2 Real and Complex Zeros of a Polynomial Function

Confirm that the third-degree polynomial function

$$f(x) = x^3 + 4x$$

has exactly three zeros: $x = 0$, $x = 2i$, and $x = -2i$.

Solution

Factor the polynomial completely as $x(x - 2i)(x + 2i)$. So, the zeros are

$$x(x - 2i)(x + 2i) = 0$$

$$x = 0$$

$$x - 2i = 0 \quad \Rightarrow \quad x = 2i$$

$$x + 2i = 0 \quad \Rightarrow \quad x = -2i.$$

In the graph in Figure 3.39, only the real zero $x = 0$ appears as an x-intercept.

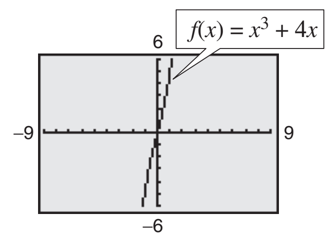


Figure 3.39

CHECKPOINT Now try Exercise 3.

Example 3 shows how to use the methods described in Sections 3.2 and 3.3 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including complex zeros.

Example 3 Finding the Zeros of a Polynomial Function

You may want to remind students that a graphing utility is helpful for determining real zeros, which in turn are useful in finding complex zeros.

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all the zeros of f.

Solution

The possible rational zeros are ± 1, ± 2, ± 4, and ± 8. The graph shown in Figure 3.40 indicates that 1 and -2 are likely zeros, and that 1 is possibly a repeated zero because it appears that the graph touches (but does not cross) the x-axis at this point. Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8 = (x - 1)(x - 1)(x + 2)(x^2 + 4).$$

By factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) = (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1, x = 1, x = -2, x = 2i, \text{ and } x = -2i$$

Note from the graph of f shown in Figure 3.40 that the *real* zeros are the only ones that appear as x-intercepts.

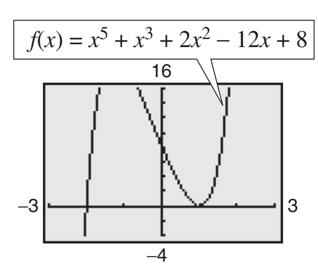


Figure 3.40

CHECKPOINT Now try Exercise 27.

Conjugate Pairs

In Example 3, note that the two complex zeros are **conjugates.** That is, they are of the forms $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients*. For instance, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

Example 4 Finding a Polynomial with Given Zeros

Find a *fourth-degree* polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9) = x^4 + 2x^3 + 10x^2 + 18x + 9.$$

CHECKPOINT Now try Exercise 39.

Factoring a Polynomial

The Linear Factorization Theorem states that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following theorem states that even if you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

Factors of a Polynomial (See the proof on page 332.)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be **prime** or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals*. For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals, but *reducible* over the reals.

STUDY TIP

Recall that irrational and rational numbers are subsets of the set of real numbers, and the real numbers are a subset of the set of complex numbers.

Example 5 Factoring a Polynomial

Write the polynomial

$$f(x) = x^4 - x^2 - 20$$

a. as the product of factors that are irreducible over the *rationals*,

b. as the product of linear factors and quadratic factors that are irreducible over the *reals*, and

c. in completely factored form.

Solution

a. Begin by factoring the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x - 2i)(x + 2i).$$

CHECKPOINT Now try Exercise 47.

In Example 5, notice from the completely factored form that the *fourth*-degree polynomial has *four* zeros.

Throughout this chapter, the results and theorems have been stated in terms of zeros of polynomial functions. Be sure you see that the same results could have been stated in terms of solutions of polynomial equations. This is true because the zeros of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

are precisely the solutions of the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0 = 0.$$

Activities

1. Write as a product of linear factors: $f(x) = x^4 - 16$.
   Answer: $(x - 2)(x + 2)(x - 2i)(x + 2i)$
2. Find a third-degree polynomial with integer coefficients that has 2 and $3 - i$ as zeros.
   Answer: $x^3 - 8x^2 + 22x - 20$
3. Write the polynomial $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 6$ in completely factored form. (*Hint:* One factor is $x^2 - 2x - 2$.)
   Answer: $\left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right)\left(x - 1 + \sqrt{2}i\right)\left(x - 1 - \sqrt{2}i\right)$

Example 6 Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \qquad \text{and} \qquad x - (1 - 3i)$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned} [x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10. \end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r} x^2 - x - 6 \\ x^2 - 2x + 10 \overline{\big) x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \phantom{{}+2x-60} \\ -x^3 - 4x^2 + 2x \phantom{{}-60} \\ \underline{-x^3 + 2x^2 - 10x} \phantom{{}-60} \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

So, you have

$$\begin{aligned} f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2) \end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

CHECKPOINT Now try Exercise 53.

Graphical Solution

Because complex zeros always occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. Because the polynomial is a fourth-degree polynomial, you know that there are at most two other zeros of the function. Use a graphing utility to graph

$$y = x^4 - 3x^3 + 6x^2 + 2x - 60$$

as shown in Figure 3.41.

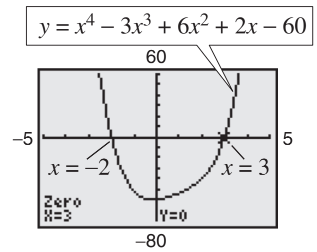


Figure 3.41

You can see that -2 and 3 appear to be x-intercepts of the graph of the function. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to confirm that $x = -2$ and $x = 3$ are x-intercepts of the graph. So, you can conclude that the zeros of f are

$$x = 1 + 3i, \; x = 1 - 3i, \; x = 3, \text{ and } x = -2.$$

In Example 6, if you were not told that $1 + 3i$ is a zero of f, you could still find all zeros of the function by using synthetic division to find the real zeros -2 and 3. Then, you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

3.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The _______ of _______ states that if $f(x)$ is a polynomial function of degree n $(n > 0)$, then f has at least one zero in the complex number system.

2. The _______ states that if $f(x)$ is a polynomial of degree n, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \cdot \cdot \cdot, c_n$ are complex numbers.

3. A quadratic factor that cannot be factored further as a product of linear factors containing real numbers is said to be _______ over the _______ .

4. If $a + bi$ is a complex zero of a polynomial with real coefficients, then so is its _______ .

In Exercises 1–4, find all the zeros of the function.

1. $f(x) = x^2(x + 3)$

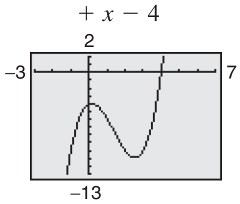
2. $g(x) = (x - 2)(x + 4)^3$

3. $f(x) = (x + 9)(x + 4i)(x - 4i)$

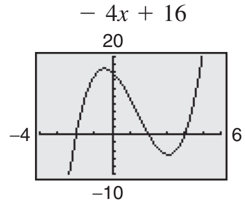
4. $h(t) = (t - 3)(t - 2)(t - 3i)(t + 3i)$

Graphical and Analytical Analysis **In Exercises 5–8, find all the zeros of the function. Is there a relationship between the number of real zeros and the number of *x*-intercepts of the graph? Explain.**

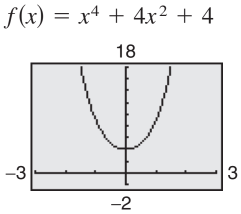
5. $f(x) = x^3 - 4x^2 + x - 4$



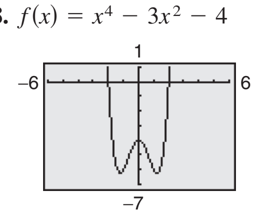
6. $f(x) = x^3 - 4x^2 - 4x + 16$



7. $f(x) = x^4 + 4x^2 + 4$



8. $f(x) = x^4 - 3x^2 - 4$



In Exercises 9–28, find all the zeros of the function and write the polynomial as a product of linear factors. Use a graphing utility to graph the function to verify your results graphically. (If possible, use your graphing utility to verify the complex zeros.)

9. $h(x) = x^2 - 4x + 1$

10. $g(x) = x^2 + 10x + 23$

11. $f(x) = x^2 - 12x + 26$

12. $f(x) = x^2 + 6x - 2$

13. $f(x) = x^2 + 25$

14. $f(x) = x^2 + 36$

15. $f(x) = 16x^4 - 81$

16. $f(y) = 81y^4 - 625$

17. $f(z) = z^2 - z + 56$

18. $h(x) = x^2 - 4x - 3$

19. $f(x) = x^4 + 10x^2 + 9$

20. $f(x) = x^4 + 29x^2 + 100$

21. $f(x) = 3x^3 - 5x^2 + 48x - 80$

22. $f(x) = 3x^3 - 2x^2 + 75x - 50$

23. $f(t) = t^3 - 3t^2 - 15t + 125$

24. $f(x) = x^3 + 11x^2 + 39x + 29$

25. $f(x) = 5x^3 - 9x^2 + 28x + 6$

26. $f(s) = 3s^3 - 4s^2 + 8s + 8$

27. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

28. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

In Exercises 29–36, (a) find all zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the *x*-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only *x*-intercepts.

29. $f(x) = x^2 - 14x + 46$

30. $f(x) = x^2 - 12x + 34$

31. $f(x) = 2x^3 - 3x^2 + 8x - 12$

32. $f(x) = 2x^3 - 5x^2 + 18x - 45$

33. $f(x) = x^3 - 11x + 150$

34. $f(x) = x^3 + 10x^2 + 33x + 34$

35. $f(x) = x^4 + 25x^2 + 144$

36. $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

In Exercises 37–42, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

37. $2, i, -i$

38. $3, 4i, -4i$

39. $2, 2, 4 - i$

40. $-1, -1, 2 + 5i$

41. $0, -5, 1 + \sqrt{2}i$

42. $0, 4, 1 + \sqrt{2}i$

In Exercises 43–46, the degree, the zeros, and a solution point of a polynomial function *f* are given. Write *f* (a) in completely factored form and (b) in expanded form.

	Degree	*Zeros*	*Solution Point*
43.	4	$1, -2, 2i$	$f(-1) = 10$
44.	4	$-1, 2, i$	$f(1) = 8$
45.	3	$-1, 2 + \sqrt{5}i$	$f(-2) = 42$
46.	3	$-2, 2 + 2\sqrt{2}i$	$f(-1) = -34$

In Exercises 47–50, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

47. $f(x) = x^4 - 6x^2 - 7$

48. $f(x) = x^4 + 6x^2 - 27$

49. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

50. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$

(*Hint:* One factor is $x^2 + 4$.)

In Exercises 51–58, use the given zero to find all the zeros of the function.

	Function	*Zero*
51.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
52.	$f(x) = x^3 + x^2 + 9x + 9$	$3i$
53.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
54.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
55.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
56.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$
57.	$h(x) = 8x^3 - 14x^2 + 18x - 9$	$\frac{1}{2}(1 - \sqrt{5}i)$
58.	$f(x) = 25x^3 - 55x^2 - 54x - 18$	$\frac{1}{5}(-2 + \sqrt{2}i)$

Graphical Analysis **In Exercises 59–62, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places and (b) find the exact values of the remaining zeros.**

59. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

60. $f(x) = x^3 + 4x^2 + 14x + 20$

61. $h(x) = 8x^3 - 14x^2 + 18x - 9$

62. $f(x) = 25x^3 - 55x^2 - 54x - 18$

63. ***Height*** A baseball is thrown upward from ground level with an initial velocity of 48 feet per second, and its height h (in feet) is given by

$$h(t) = -16t^2 + 48t, \quad 0 \le t \le 3$$

where t is the time (in seconds). You are told that the ball reaches a height of 64 feet. Is this possible? Explain.

64. ***Profit*** The demand equation for a microwave is $p = 140 - 0.0001x$, where p is the unit price (in dollars) of the microwave and x is the number of units produced and sold. The cost equation for the microwave is $C = 80x + 150{,}000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit obtained by producing and selling x units is given by $P = R - C = xp - C$. You are working in the marketing department of the company that produces this microwave, and you are asked to determine a price p that would yield a profit of \$9 million. Is this possible? Explain.

Synthesis

True or False? **In Exercises 65 and 66, decide whether the statement is true or false. Justify your answer.**

65. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

66. If $x = 4 + 3i$ is a zero of the function

$$f(x) = x^4 - 7x^3 - 13x^2 + 265x - 750$$

then $x = 4 - 3i$ must also be a zero of f.

67. ***Exploration*** Use a graphing utility to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of k. Find values of k such that the zeros of f satisfy the specified characteristics. (Some parts have many correct answers.)

(a) Two real zeros, each of multiplicity 2

(b) Two real zeros and two complex zeros

68. ***Writing*** Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

Skills Review

In Exercises 69–72, sketch the graph of the quadratic function. Identify the vertex and any intercepts. Use a graphing utility to verify your results.

69. $f(x) = x^2 - 7x - 8$

70. $f(x) = -x^2 + x + 6$

71. $f(x) = 6x^2 + 5x - 6$

72. $f(x) = 4x^2 + 2x - 12$