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P.2 Exponents and Radicals

Integer Exponents

Repeated *multiplication* can be written in **exponential form.**

Repeated Multiplication	*Exponential Form*
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

In general, if a is a real number, variable, or algebraic expression and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read "a to the nth **power.**" An exponent can be negative as well. Property 3 below shows how to use a negative exponent.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	*Example*
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert(-2)^2\rvert = \lvert -2 \rvert^2 = 2^2 = 4$

It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$, whereas $-2^4 = -16$. It is also important to know when to use parentheses when evaluating exponential expressions using a graphing calculator. Figure P.9 shows that a graphing calculator follows the order of operations.

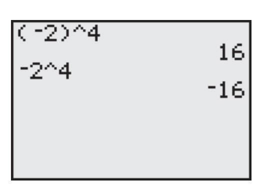

Figure P.9

What you should learn

- Use properties of exponents.
- Use scientific notation to represent real numbers.
- Use properties of radicals.
- Simplify and combine radicals.
- Rationalize denominators and numerators.
- Use properties of rational exponents.

Why you should learn it

Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 115 on page 23, you will use an expression involving a radical to find the size of a particle that can be carried by a stream moving at a certain velocity.

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The properties of exponents listed on the preceding page apply to *all* integers m and n, not just positive integers. For instance, by Property 2, you can write

$$\frac{3^4}{3^{-5}} = 3^{4-(-5)} = 3^{4+5} = 3^9.$$

Example 1 Using Properties of Exponents

a. $(-3ab^4)(4ab^{-3}) = -12(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \; a \neq 0$

CHECKPOINT Now try Exercise 15.

Example 2 Rewriting with Positive Exponents

a. $x^{-1} = \dfrac{1}{x}$ — Property 3

b. $\dfrac{1}{3x^{-2}} = \dfrac{1(x^2)}{3} = \dfrac{x^2}{3}$ — The exponent -2 does not apply to 3.

c. $\dfrac{1}{(3x)^{-2}} = (3x)^2 = 9x^2$ — The exponent -2 does apply to 3.

d. $\dfrac{12a^3b^{-4}}{4a^{-2}b} = \dfrac{12a^3 \cdot a^2}{4b \cdot b^4} = \dfrac{3a^5}{b^5}$ — Properties 3 and 1

e. $\left(\dfrac{3x^2}{y}\right)^{-2} = \dfrac{3^{-2}(x^2)^{-2}}{y^{-2}}$ — Properties 5 and 7

$= \dfrac{3^{-2}x^{-4}}{y^{-2}}$ — Property 6

$= \dfrac{y^2}{3^2x^4} = \dfrac{y^2}{9x^4}$ — Property 3, and simplify.

CHECKPOINT Now try Exercise 19.

Example 3 Calculators and Exponents

	Expression	Graphing Calculator Keystrokes	Display
a.	$3^{-2} + 4^{-1}$	3 [^] [(-)] 2 [+] 4 [^] [(-)] 1 [ENTER]	.3611111111
b.	$\dfrac{3^5 + 1}{3^5 - 1}$	[(] 3 [^] 5 [+] 1 [)] [÷] [(] 3 [^] 5 [−] 1 [)] [ENTER]	1.008264463

CHECKPOINT Now try Exercise 23.

TECHNOLOGY TIP The graphing calculator keystrokes given in this text may not be the same as the keystrokes for your graphing calculator. Be sure you are familiar with the use of the keys on your own calculator.

STUDY TIP

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented in this text. The important thing is to use steps that you understand *and*, of course, that are justified by the rules of algebra. For instance, you might prefer the following steps for Example 2(e).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Additional Examples

a. $4x^{-1} = \dfrac{4}{x}$

b. $\dfrac{3}{2x^{-2}} = \dfrac{3x^2}{2}$

c. $\left(\dfrac{2}{5}\right)^{-1} = \dfrac{5}{2}$

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Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 359 billion billion gallons of water on Earth—that is, 359 followed by 18 zeros.

$$359{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \le c < 10$ and n is an integer. So, the number of gallons of water on Earth can be written in scientific notation as

$$3.59 \times 100{,}000{,}000{,}000{,}000{,}000{,}000 = 3.59 \times 10^{20}.$$

The *positive* exponent 20 indicates that the number is *large* (10 or more) and that the decimal point has been moved 20 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.0000000000000000000000000009.$$

28 decimal places

Activities

1. Simplify: $\left(\dfrac{6x^{-1}y}{y^3}\right)^{-2}$.

   Answer: $\dfrac{x^2y^4}{36}$

2. Write in scientific notation: 39,000,000.

   Answer: 3.9×10^7

3. Write in decimal notation: 5.312×10^{-5}.

   Answer: 0.00005312

Example 4 Scientific Notation

a. $1.345 \times 10^2 = 134.5$

b. $0.0000782 = 7.82 \times 10^{-5}$

c. $-9.36 \times 10^{-6} = -0.00000936$

d. $836{,}100{,}000 = 8.361 \times 10^8$

CHECKPOINT Now try Exercise 31.

TECHNOLOGY TIP Most calculators automatically switch to scientific notation when they are showing large or small numbers that exceed the display range. Try evaluating $86{,}500{,}000 \times 6000$. If your calculator follows standard conventions, its display should be

5.19 11 or 5.19 E 11

which is 5.19×10^{11}.

Example 5 Using Scientific Notation with a Calculator

Use a calculator to evaluate $65{,}000 \times 3{,}400{,}000{,}000$.

Solution

Because $65{,}000 = 6.5 \times 10^4$ and $3{,}400{,}000{,}000 = 3.4 \times 10^9$, you can multiply the two numbers using the following graphing calculator keystrokes.

6.5 [EE] 4 [×] 3.4 [EE] 9 [ENTER]

After entering these keystrokes, the calculator display should read [2.21 E 14]. So, the product of the two numbers is

$$(6.5 \times 10^4)(3.4 \times 10^9) = 2.21 \times 10^{14} = 221{,}000{,}000{,}000{,}000.$$

CHECKPOINT Now try Exercise 53.

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Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of $25 = 5 \cdot 5$. In a similar way, a **cube root** of a number is one of its three equal factors, as in $125 = 5^3$.

Use pattern recognition to help students identify perfect squares, cubes, etc., of both positive and negative integers when simplifying radicals. Have students construct a table of powers for several integers. For example:

n	n^2	n^3	n^4
-3	9	-27	81
-2	4	-8	16
-1	1	-1	1
0	0	0	0
1	1	1	1
2	4	8	16
3	9	27	81

Definition of the *n*th Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** If $n = 2$, the root is a **square root.** If $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The *principal square root* of 25, written as $\sqrt{25}$, is the positive root, 5. The **principal *n*th root** of a number is defined as follows.

Principal *n*th Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}. \qquad \text{Principal } n\text{th root}$$

The positive integer n is the **index** of the radical, and the number a is the **radicand.** If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices*.)

A common misunderstanding when taking square roots of real numbers is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: ~~$\sqrt{4} = \pm 2$~~ Correct: $-\sqrt{4} = -2$ and $\sqrt{4} = 2$

Example 6 Evaluating Expressions Involving Radicals

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-\left(\sqrt{36}\right) = -\left(\sqrt{6^2}\right) = -(6) = -6$.

c. $\sqrt[3]{\frac{125}{64}} = \frac{5}{4}$ because $\left(\frac{5}{4}\right)^3 = \frac{5^3}{4^3} = \frac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

CHECKPOINT Now try Exercise 59.

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Here are some generalizations about the nth roots of a real number.

Generalizations About nth Roots of Real Numbers

Real number a	Integer n	Root(s) of a	Example
$a > 0$	$n > 0$, n is even.	$\sqrt[n]{a}, -\sqrt[n]{a}$	$\sqrt[4]{81} = 3, -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd.	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even.	No real roots	$\sqrt{-4}$ is not a real number.
$a = 0$	n is even or odd.	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

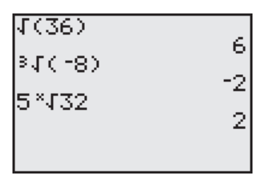
Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	*Example*
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}, \quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

TECHNOLOGY TIP

There are three methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* $\boxed{\sqrt{\ }}$. For cube roots, you can use the *cube root key* $\boxed{\sqrt[3]{\ }}$ (or menu choice). For other roots, you can use the *xth root key* $\boxed{\sqrt[x]{\ }}$ (or menu choice). For example, the screen below shows you how to evaluate $\sqrt{36}$, $\sqrt[3]{-8}$, and $\sqrt[5]{32}$ using one of the three methods described.



Example 7 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt{8} \cdot \sqrt{2}$ **b.** $\left(\sqrt[3]{5}\right)^3$ **c.** $\sqrt[3]{x^3}$ **d.** $\sqrt[6]{y^6}$

Solution

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$

b. $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$

d. $\sqrt[6]{y^6} = \lvert y \rvert$

CHECKPOINT Now try Exercise 79.

Additional Examples

a. $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$

b. $\sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{\sqrt{4}} = \dfrac{\sqrt{3}}{2}$

c. $\left(\sqrt{10}\right)^2 = 10$

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Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

Example 8 Simplifying Even Roots

Perfect 4th power; Leftover factor

a. $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

Perfect square; Leftover factor

b. $\sqrt{75x^3} = \sqrt{25x^2 \cdot 3x}$ Find largest square factor.

$= \sqrt{(5x)^2 \cdot 3x}$

$= 5x\sqrt{3x}$ Find root of perfect square.

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

CHECKPOINT Now try Exercise 81(a).

Example 9 Simplifying Odd Roots

Perfect cube; Leftover factor

a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

Perfect cube; Leftover factor

b. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5}$ Find largest cube factor.

$= \sqrt[3]{(-2x^2)^3 \cdot 5}$

$= -2x^2\sqrt[3]{5}$ Find root of perfect cube.

CHECKPOINT Now try Exercise 81(b).

Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

STUDY TIP

When you simplify a radical, it is important that both expressions are defined for the same values of the variable. For instance, in Example 8(b), $\sqrt{75x^3}$ and $5x\sqrt{3x}$ are both defined only for nonnegative values of x. Similarly, in Example 8(c), $\sqrt[4]{(5x)^4}$ and $5|x|$ are both defined for all real values of x.

Example 10 Combining Radicals

a. $2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3}$ — Find square factors.

$= 8\sqrt{3} - 9\sqrt{3}$ — Find square roots and multiply by coefficients.

$= (8 - 9)\sqrt{3}$ — Combine like terms.

$= -\sqrt{3}$ — Simplify.

b. $\sqrt[3]{16x} - \sqrt[3]{54x^4} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x}$ — Find cube factors.

$= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x}$ — Find cube roots.

$= (2 - 3x)\sqrt[3]{2x}$ — Combine like terms.

CHECKPOINT Now try Exercise 85.

Try using your calculator to check the result of Example 10(a). You should obtain -1.732050808, which is the same as the calculator's approximation for $-\sqrt{3}$.

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$. Note that the product of a number and its conjugate is a rational number.

Example 11 Rationalizing Denominators

Rationalize the denominator of each expression.

a. $\dfrac{5}{2\sqrt{3}}$ **b.** $\dfrac{2}{\sqrt[3]{5}}$

Solution

a. $\dfrac{5}{2\sqrt{3}} = \dfrac{5}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ — $\sqrt{3}$ is rationalizing factor.

$= \dfrac{5\sqrt{3}}{2(3)}$ — Multiply.

$= \dfrac{5\sqrt{3}}{6}$ — Simplify.

b. $\dfrac{2}{\sqrt[3]{5}} = \dfrac{2}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$ — $\sqrt[3]{5^2}$ is rationalizing factor.

$= \dfrac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}} = \dfrac{2\sqrt[3]{25}}{5}$ — Multiply and simplify.

CHECKPOINT Now try Exercise 91.

STUDY TIP

Notice in Example 11(b) that the numerator and denominator are multiplied by $\sqrt[3]{5^2}$ to produce a perfect cube radicand.

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Example 12 Rationalizing a Denominator with Two Terms

Rationalize the denominator of $\dfrac{2}{3 + \sqrt{7}}$.

Solution

$$\frac{2}{3+\sqrt{7}} = \frac{2}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}}$$ Multiply numerator and denominator by conjugate of denominator.

$$= \frac{2\left(3-\sqrt{7}\right)}{(3)^2 - \left(\sqrt{7}\right)^2}$$ Find products. In denominator, $(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$.

$$= \frac{2\left(3-\sqrt{7}\right)}{2} = 3 - \sqrt{7}$$ Simplify and divide out common factors.

CHECKPOINT Now try Exercise 93.

In calculus, sometimes it is necessary to rationalize the numerator of an expression.

Example 13 Rationalizing a Numerator

Rationalize the numerator of $\dfrac{\sqrt{5} - \sqrt{7}}{2}$.

Solution

$$\frac{\sqrt{5}-\sqrt{7}}{2} = \frac{\sqrt{5}-\sqrt{7}}{2} \cdot \frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$ Multiply numerator and denominator by conjugate of numerator.

$$= \frac{\left(\sqrt{5}\right)^2 - \left(\sqrt{7}\right)^2}{2\left(\sqrt{5}+\sqrt{7}\right)}$$ Find products. In numerator, $(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$.

$$= \frac{-2}{2\left(\sqrt{5}+\sqrt{7}\right)} = \frac{-1}{\sqrt{5}+\sqrt{7}}$$ Simplify and divide out common factors.

CHECKPOINT Now try Exercise 97.

Rational Exponents

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as

$$a^{1/n} = \sqrt[n]{a} \text{ where } 1/n \text{ is the } \textbf{rational exponent} \text{ of } a.$$

Moreover, if m is a positive integer that has no common factor with n, then

$$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m \quad \text{and} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}.$$

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

Additional Examples

a. $\dfrac{3}{\sqrt{7}} = \dfrac{3}{\sqrt{7}} \cdot \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{3\sqrt{7}}{7}$

b. $\dfrac{2}{\sqrt[3]{4}} = \dfrac{2}{\sqrt[3]{4}} \cdot \dfrac{\sqrt[3]{2}}{\sqrt[3]{2}}$

$= \dfrac{2\sqrt[3]{2}}{2} = \sqrt[3]{2}$

c. $\dfrac{6}{\sqrt{2}+\sqrt{3}}$

$= \dfrac{6}{\sqrt{2}+\sqrt{3}} \cdot \dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}$

$= \dfrac{6\left(\sqrt{2}-\sqrt{3}\right)}{-1}$

$= -6\sqrt{2} + 6\sqrt{3}$

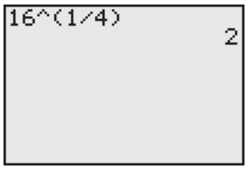
STUDY TIP

Do not confuse the expression $\sqrt{5} + \sqrt{7}$ with the expression $\sqrt{5+7}$. In general, $\sqrt{x+y}$ does not equal $\sqrt{x} + \sqrt{y}$. Similarly, $\sqrt{x^2 + y^2}$ does not equal $x + y$.

Rationalizing the numerator is especially useful when finding limits in calculus.

TECHNOLOGY TIP

Another method of evaluating radicals on a graphing calculator involves converting the radical to exponential form and then using the *exponential key* ^. Be sure to use parentheses around the rational exponent. For example, the screen below shows you how to evaluate $\sqrt[4]{16}$.



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The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken.

Power

Index

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

When you are working with rational exponents, the properties of integer exponents still apply. For instance,

$$2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}.$$

Example 14 Changing from Radical to Exponential Form

a. $\sqrt{3} = 3^{1/2}$

b. $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{(5/2)}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

CHECKPOINT Now try Exercise 99.

Example 15 Changing from Exponential to Radical Form

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$

d. $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

CHECKPOINT Now try Exercise 101.

Rational exponents are useful for evaluating roots of numbers on a calculator, reducing the index of a radical, and simplifying calculus expressions.

Example 16 Simplifying with Rational Exponents

a. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

b. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \qquad x \neq 0$

c. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$

d. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

e. $(2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{(4/3)-(1/3)} = 2x - 1, \qquad x \neq \dfrac{1}{2}$

CHECKPOINT Now try Exercise 107.

STUDY TIP

Rational exponents can be tricky, and you must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual-looking results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

Additional Examples

a. $\dfrac{9^{2/3}}{9^{1/6}} = 9^{(2/3)-(1/6)}$
$= 9^{1/2}$
$= 3$

b. $\sqrt[3]{6} + \sqrt[3]{48} = \sqrt[3]{6} + \sqrt[3]{8 \cdot 6}$
$= \sqrt[3]{6} + 2\sqrt[3]{6}$
$= 3\sqrt[3]{6}$

Activities

1. Simplify: $\sqrt[3]{250x^6y^4}$.
   Answer: $5x^2y\sqrt[3]{2y}$
2. Evaluate: $\sqrt[3]{64^2}$.
   Answer: 16
3. Are the two expressions equivalent?
   $\dfrac{2}{\sqrt{3} - 1}, \qquad \sqrt{3} + 1$
   Answer: Yes

STUDY TIP

The expression in Example 16(e) is not defined when $x = \frac{1}{2}$ because

$$\left(2 \cdot \tfrac{1}{2} - 1\right)^{-1/3} = (0)^{-1/3}$$

is not a real number.

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P.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. In the exponential form a^n, n is the _______ and a is the _______ .
2. A convenient way of writing very large or very small numbers is called _______ .
3. One of the two equal factors of a number is called a _______ of the number.
4. The _______ of a number is the nth root that has the same sign as a, and is denoted by $\sqrt[n]{a}$.
5. In the radical form $\sqrt[n]{a}$, the positive integer n is called the _______ of the radical and the number a is called the _______ .
6. When an expression involving radicals has all possible factors removed, radical-free denominators, and a reduced index, it is in _______.
7. The expressions $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are _______ of each other.
8. The process used to create a radical-free denominator is known as _______ the denominator.
9. In the expression $b^{m/n}$, m denotes the _______ to which the base is raised and n denotes the _______ or root to be taken.

In Exercises 1–8, evaluate each expression.

1. (a) $4^2 \cdot 3$ (b) $3 \cdot 3^3$
2. (a) $\dfrac{5^5}{5^2}$ (b) $\dfrac{3^2}{3^4}$
3. (a) $(3^3)^2$ (b) -3^2
4. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$
5. (a) $\dfrac{3}{3^{-4}}$ (b) $24(-2)^{-5}$
6. (a) $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$
7. (a) $2^{-1} + 3^{-1}$ (b) $(2^{-1})^{-2}$
8. (a) $3^{-1} + 2^{-2}$ (b) $(3^{-2})^2$

In Exercises 9–14, evaluate the expression for the value of x.

	Expression	*Value*
9.	$7x^{-2}$	2
10.	$6x^0 - (6x)^0$	7
11.	$2x^3$	-3
12.	$-3x^4$	-2
13.	$4x^2$	$-\frac{1}{2}$
14.	$5(-x)^3$	$\frac{1}{3}$

In Exercises 15–20, simplify each expression.

15. (a) $(-5z)^3$ (b) $5x^4(x^2)$
16. (a) $(3x)^2$ (b) $(4x^3)^2$
17. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x + y)^3}{9(x + y)}$
18. (a) $\dfrac{r^4}{r^6}$ (b) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$
19. (a) $[(x^2y^{-2})^{-1}]^{-1}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$
20. (a) $(2x^5)^0, \quad x \neq 0$ (b) $(5x^2z^6)^3(5x^2z^6)^{-3}$

In Exercises 21–24, use a calculator to evaluate the expression. (Round your answer to three decimal places.)

21. $(-4)^3(5^2)$
22. $(8^{-4})(10^3)$
23. $\dfrac{3^6}{7^3}$
24. $\dfrac{4^3}{3^{-4}}$

In Exercises 25–34, write the number in scientific notation.

25. 852.25
26. 28,022.2
27. 10,252.484
28. 525,252,118
29. -1110.25
30. $-5,222,145$
31. 0.0002485
32. 0.0000025
33. -0.0000025
34. -0.000125005

In Exercises 35–42, write the number in decimal notation.

35. 1.25×10^5
36. 1.08×10^4
37. -4.816×10^8
38. -3.785×10^{10}
39. 3.25×10^{-8}
40. 5.05×10^{-10}
41. -9.001×10^{-3}
42. -8.098×10^{-6}

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In Exercises 43–46, write the number in scientific notation.

43. Land area of Earth: 57,300,000 square miles

44. Light year: 9,460,000,000,000 kilometers

45. Relative density of hydrogen: 0.0000899 gram per cubic centimeter

46. One micron (millionth of a meter): 0.00003937 inch

In Exercises 47–50, write the number in decimal notation.

47. Daily consumption of Coca-Cola products worldwide: 5.71×10^8 drinks (Source: The Coca-Cola Company)

48. Interior temperature of sun: 1.5×10^7 degrees Celsius

49. Charge of electron: 1.6022×10^{-19} coulomb

50. Width of human hair: 9.0×10^{-5} meter

In Exercises 51 and 52, evaluate the expression without using a calculator.

51. $\sqrt{25 \times 10^8}$ **52.** $\sqrt[3]{8 \times 10^{15}}$

In Exercises 53–56, use a calculator to evaluate each expression. (Round your answer to three decimal places.)

53. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$

(b) $\dfrac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

54. (a) $750\left(1 + \dfrac{0.11}{365}\right)^{800}$

(b) $\dfrac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

55. (a) $\sqrt{4.5 \times 10^9}$ (b) $\sqrt[3]{6.3 \times 10^4}$

56. (a) $(2.65 \times 10^{-4})^{1/3}$ (b) $\sqrt{9 \times 10^{-4}}$

In Exercises 57–66, evaluate the expression without using a calculator.

57. $\sqrt{121}$ **58.** $\sqrt{16}$

59. $-\sqrt[3]{-27}$ **60.** $\dfrac{\sqrt[4]{81}}{3}$

61. $\left(\sqrt[3]{-125}\right)^3$ **62.** $\sqrt[4]{562^4}$

63. $32^{-3/5}$ **64.** $\left(\frac{9}{4}\right)^{-1/2}$

65. $\left(-\dfrac{1}{64}\right)^{-1/3}$ **66.** $-\left(\dfrac{1}{125}\right)^{-4/3}$

In Exercises 67–78, use a calculator to approximate the value of the expression. (Round your answer to three decimal places.)

67. $\sqrt[5]{-27^3}$ **68.** $\sqrt[3]{45^2}$

69. $(3.4)^{2.5}$ **70.** $(6.1)^{-2.9}$

71. $(1.2^{-2})\sqrt{75} + 3\sqrt{8}$ **72.** $\dfrac{-5 + \sqrt{33}}{5}$

73. $\sqrt{\pi + 1}$ **74.** $\sqrt{10 - \pi}$

75. $\dfrac{3.14}{\pi} + \sqrt[3]{5}$ **76.** $\dfrac{\sqrt{10}}{2.5} - \pi^2$

77. $(2.8)^{-2} + 1.01 \times 10^6$

78. $2.12 \times 10^{-2} + \sqrt{15}$

In Exercises 79 and 80, use the properties of radicals to simplify each expression.

79. (a) $\left(\sqrt[4]{3}\right)^4$ (b) $\sqrt[5]{96x^5}$

80. (a) $\sqrt{12} \cdot \sqrt{3}$ (b) $\sqrt[4]{x^4}$

In Exercises 81–86, simplify each expression.

81. (a) $\sqrt{54xy^4}$ (b) $\sqrt[3]{\dfrac{32a^2}{b^2}}$

82. (a) $\sqrt[3]{54}$ (b) $\sqrt{32x^3y^4}$

83. (a) $2\sqrt{50} + 12\sqrt{8}$ (b) $10\sqrt{32} - 6\sqrt{18}$

84. (a) $5\sqrt{x} - 3\sqrt{x}$

(b) $-2\sqrt{9y} + 10\sqrt{y}$

85. (a) $3\sqrt{x + 1} + 10\sqrt{x + 1}$

(b) $7\sqrt{80x} - 2\sqrt{125x}$

86. (a) $5\sqrt{10x^2} - \sqrt{90x^2}$

(b) $8\sqrt[3]{27x} - \frac{1}{2}\sqrt[3]{64x}$

In Exercises 87–90, complete the statement with <, =, or >.

87. $\sqrt{5} + \sqrt{3} \;\square\; \sqrt{5 + 3}$ **88.** $\sqrt{\dfrac{3}{11}} \;\square\; \dfrac{\sqrt{3}}{\sqrt{11}}$

89. $5 \;\square\; \sqrt{3^2 + 2^2}$ **90.** $5 \;\square\; \sqrt{3^2 + 4^2}$

In Exercises 91–94, rationalize the denominator of the expression. Then simplify your answer.

91. $\dfrac{1}{\sqrt{3}}$ **92.** $\dfrac{8}{\sqrt[3]{2}}$

93. $\dfrac{5}{\sqrt{14} - 2}$ **94.** $\dfrac{3}{\sqrt{5} + \sqrt{6}}$

In Exercises 95–98, rationalize the numerator of the expression. Then simplify your answer.

95. $\dfrac{\sqrt{8}}{2}$ **96.** $\dfrac{\sqrt{2}}{3}$

97. $\dfrac{\sqrt{5} + \sqrt{3}}{3}$ **98.** $\dfrac{\sqrt{7} - 3}{4}$

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

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In Exercises 99–106, fill in the missing form of the expression.

	Radical Form	*Rational Exponent Form*
99.	$\sqrt[3]{64}$	
100.		$-(144^{1/2})$
101.		$32^{1/5}$
102.	$\sqrt[3]{614.125}$	
103.	$\sqrt[3]{-216}$	
104.		$(-243)^{1/5}$
105.	$\sqrt[4]{81^3}$	
106.		$16^{5/4}$

In Exercises 107–110, perform the operations and simplify.

107. $\dfrac{(2x^2)^{3/2}}{2^{1/2}x^4}$ **108.** $\dfrac{x^{4/3}y^{2/3}}{(xy)^{1/3}}$

109. $\dfrac{x^{-3} \cdot x^{1/2}}{x^{3/2} \cdot x^{-1}}$ **110.** $\dfrac{5^{-1/2} \cdot 5x^{5/2}}{(5x)^{3/2}}$

In Exercises 111 and 112, reduce the index of each radical and rewrite in radical form.

111. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x+1)^4}$

112. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

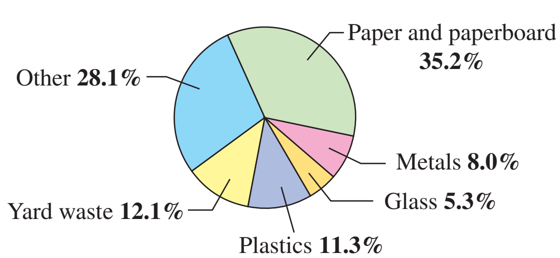
In Exercises 113 and 114, write each expression as a single radical. Then simplify your answer.

113. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$

114. (a) $\sqrt{\sqrt{243(x+1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

115. ***Erosion*** A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size of the particle that can be carried by a stream flowing at the rate of $\frac{3}{4}$ foot per second.

116. ***Environment*** There was 2.362×10^8 tons of municipal waste generated in 2003. Find the number of tons for each of the categories in the graph. (Source: Franklin Associates, a Division of ERG)



117. ***Tropical Storms*** The table shows the number of Atlantic tropical storms and hurricanes per year from 1995 to 2005. Find the average number of tropical storms and hurricanes from 1995 to 2005. Is your answer an *integer,* a *rational number,* or an *irrational number*? Explain. (Source: NOAA)

Year	Number of tropical storms and hurricanes
1995	19
1996	13
1997	8
1998	14
1999	12
2000	15
2001	15
2002	12
2003	16
2004	15
2005	27

118. ***Mathematical Modeling*** A funnel is filled with water to a height of h centimeters. The formula

$$t = 0.03[12^{5/2} - (12 - h)^{5/2}], \quad 0 \le h \le 12$$

represents the amount of time t (in seconds) it will take for the funnel to empty. Find t for $h = 7$ centimeters.

Synthesis

True or False? **In Exercises 119 and 120, determine whether the statement is true or false. Justify your answer.**

119. $\dfrac{x^{k+1}}{x} = x^k$ **120.** $(a^n)^k = a^{(n^k)}$

121. ***Think About It*** Verify that $a^0 = 1$, $a \neq 0$. (*Hint:* Use the property of exponents $a^m/a^n = a^{m-n}$.)

122. ***Think About It*** Is the real number 52.7×10^5 written in scientific notation? Explain.

123. ***Exploration*** List all possible digits that occur in the units place of the square of a positive integer. Use that list to determine whether $\sqrt{5233}$ is an integer.

124. ***Think About It*** Square the real number $2/\sqrt{5}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?