P.4 Rational Expressions

Domain of an Algebraic Expression

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain. For instance, the expressions $(x + 1) + (x + 2)$ and $2x + 3$ are equivalent because

$$(x + 1) + (x + 2) = x + 1 + x + 2 = x + x + 1 + 2 = 2x + 3.$$

What you should learn

- Find domains of algebraic expressions.
- Simplify rational expressions.
- Add, subtract, multiply, and divide rational expressions.
- Simplify complex fractions.

Why you should learn it

Rational expressions are useful in estimating the temperature of food as it cools. For instance, a rational expression is used in Exercise 96 on page 46 to model the temperature of food as it cools in a refrigerator set at 40°F.

Dwayne Newton/PhotoEdit

Example 1 Finding the Domain of an Algebraic Expression

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers, unless the domain is specifically restricted.

b. The domain of the radical expression

$$\sqrt{x - 2}$$

is the set of real numbers greater than or equal to 2, because the square root of a negative number is not a real number.

c. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, which would result in division by zero, which is undefined.

CHECKPOINT Now try Exercise 5.

The quotient of two algebraic expressions is a **fractional expression.** Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \quad \frac{2x - 1}{x + 1}, \quad \text{or} \quad \frac{x^2 - 1}{x^2 + 1}$$

is a **rational expression.**

Alert students to the importance of the domain of an expression in graphing functions later in this course and in calculus.

Simplifying Rational Expressions

Recall that a fraction is in simplest form if its numerator and denominator have no factors in common aside from ± 1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}, \qquad c \neq 0.$$

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials. When simplifying rational expressions, be sure to factor each polynomial completely before concluding that the numerator and denominator have no factors in common.

Example 2 Simplifying a Rational Expression

Write $\dfrac{x^2 + 4x - 12}{3x - 6}$ in simplest form.

Solution

$$\frac{x^2 + 4x - 12}{3x - 6} = \frac{(x + 6)\cancel{(x - 2)}}{3\cancel{(x - 2)}} \qquad \text{Factor completely.}$$

$$= \frac{x + 6}{3}, \quad x \neq 2 \qquad \text{Divide out common factors.}$$

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). To make sure that the simplified expression is *equivalent* to the original expression, you must restrict the domain of the simplified expression by excluding the value $x = 2$.

CHECKPOINT Now try Exercise 27.

It may sometimes be necessary to change the sign of a factor by factoring out (-1) to simplify a rational expression, as shown in Example 3.

Example 3 Simplifying a Rational Expression

Write $\dfrac{12 + x - x^2}{2x^2 - 9x + 4}$ in simplest form.

Solution

$$\frac{12 + x - x^2}{2x^2 - 9x + 4} = \frac{(4 - x)(3 + x)}{(2x - 1)(x - 4)} \qquad \text{Factor completely.}$$

$$= \frac{-\cancel{(x - 4)}(3 + x)}{(2x - 1)\cancel{(x - 4)}} \qquad (4 - x) = -(x - 4)$$

$$= -\frac{3 + x}{2x - 1}, \quad x \neq 4 \qquad \text{Divide out common factors.}$$

CHECKPOINT Now try Exercise 35.

Operations with Rational Expressions

To multiply or divide rational expressions, you can use the properties of fractions discussed in Section P.1. Recall that to divide fractions you invert the divisor and multiply.

STUDY TIP

In this text, when a rational expression is written, the domain is usually not listed with the expression. It is *implied* that the real numbers that make the denominator zero are excluded from the expression. Also, when performing operations with rational expressions, this text follows the convention of listing *beside the simplified expression* all values of x that must be specifically excluded from the domain in order to make the domains of the simplified and original expressions agree. In Example 3, for instance, the restriction $x \neq 4$ is listed beside the simplified expression to make the two domains agree. Note that the value $x = \frac{1}{2}$ is excluded from *both* domains, so it is not necessary to list this value.

The factoring technique used in equating $(4 - x)$ and $-(x - 4)$ in Example 3 is not always obvious to students. Identify the technique in one or two examples.

Example 4 Multiplying Rational Expressions

$$\frac{2x^2 + x - 6}{x^2 + 4x - 5} \cdot \frac{x^3 - 3x^2 + 2x}{4x^2 - 6x} = \frac{\cancel{(2x - 3)}(x + 2)}{(x + 5)\cancel{(x - 1)}} \cdot \frac{\cancel{x}(x - 2)\cancel{(x - 1)}}{2\cancel{x}\cancel{(2x - 3)}}$$

$$= \frac{(x + 2)(x - 2)}{2(x + 5)}, \quad x \neq 0, x \neq 1, x \neq \frac{3}{2}$$

CHECKPOINT Now try Exercise 51.

Example 5 Dividing Rational Expressions

Divide $\dfrac{x^3 - 8}{x^2 - 4}$ by $\dfrac{x^2 + 2x + 4}{x^3 + 8}$.

Solution

$$\frac{x^3 - 8}{x^2 - 4} \div \frac{x^2 + 2x + 4}{x^3 + 8} = \frac{x^3 - 8}{x^2 - 4} \cdot \frac{x^3 + 8}{x^2 + 2x + 4}$$

Invert and multiply.

$$= \frac{\cancel{(x - 2)}\cancel{(x^2 + 2x + 4)}}{\cancel{(x + 2)}\cancel{(x - 2)}} \cdot \frac{\cancel{(x + 2)}(x^2 - 2x + 4)}{\cancel{(x^2 + 2x + 4)}}$$

$$= x^2 - 2x + 4, \quad x \neq \pm 2$$

Divide out common factors.

CHECKPOINT Now try Exercise 53.

To add or subtract rational expressions, you can use the LCD (least common denominator) method or the basic definition

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad b \neq 0 \text{ and } d \neq 0.$$

Basic definition

This definition provides an efficient way of adding or subtracting *two* fractions that have no common factors in their denominators.

Example 6 Subtracting Rational Expressions

Subtract $\dfrac{2}{3x + 4}$ from $\dfrac{x}{x - 3}$.

Solution

$$\frac{x}{x - 3} - \frac{2}{3x + 4} = \frac{x(3x + 4) - 2(x - 3)}{(x - 3)(3x + 4)}$$

Basic definition

$$= \frac{3x^2 + 4x - 2x + 6}{(x - 3)(3x + 4)}$$

Distributive Property

$$= \frac{3x^2 + 2x + 6}{(x - 3)(3x + 4)}$$

Combine like terms.

CHECKPOINT Now try Exercise 57.

STUDY TIP

When subtracting rational expressions, remember to distribute the negative sign to *all* the terms in the quantity that is being subtracted.

For three or more fractions, or for fractions with a repeated factor in the denominators, the LCD method works well. Recall that the least common denominator of several fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. Here is a numerical example.

$$\frac{1}{6}+\frac{3}{4}-\frac{2}{3}=\frac{1\cdot 2}{6\cdot 2}+\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4}$$ The LCD is 12.

$$=\frac{2}{12}+\frac{9}{12}-\frac{8}{12}$$

$$=\frac{3}{12}=\frac{1}{4}$$

Sometimes the numerator of the answer has a factor in common with the denominator. In such cases the answer should be simplified. For instance, in the example above, $\frac{3}{12}$ was simplified to $\frac{1}{4}$.

Example 7 Combining Rational Expressions: The LCD Method

Perform the operations and simplify.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{x^2-1}$$

Solution

Using the factored denominators $(x-1)$, x, and $(x+1)(x-1)$, you can see that the LCD is $x(x+1)(x-1)$.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)}{x(x+1)(x-1)}-\frac{2(x+1)(x-1)}{x(x+1)(x-1)}+\frac{(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3(x)(x+1)-2(x+1)(x-1)+(x+3)(x)}{x(x+1)(x-1)}$$

$$=\frac{3x^2+3x-2x^2+2+x^2+3x}{x(x+1)(x-1)}$$ Distributive Property

$$=\frac{(3x^2-2x^2+x^2)+(3x+3x)+2}{x(x+1)(x-1)}$$ Group like terms.

$$=\frac{2x^2+6x+2}{x(x+1)(x-1)}$$ Combine like terms.

$$=\frac{2(x^2+3x+1)}{x(x+1)(x-1)}$$ Factor.

CHECKPOINT Now try Exercise 63.

Complex Fractions

Fractional expressions with separate fractions in the numerator, denominator, or both are called **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

A complex fraction can be simplified by combining the fractions in its numerator into a single fraction and then combining the fractions in its denominator into a single fraction. Then invert the denominator and multiply.

Example 8 Simplifying a Complex Fraction

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]}$$ Combine fractions.

$$= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)}$$ Simplify.

$$= \frac{2-3x}{x} \cdot \frac{x-1}{x-2}$$ Invert and multiply.

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

Additional Example

$$\frac{\left(\frac{x}{4}+\frac{3}{2}\right)}{\left(2-\frac{3}{x}\right)} = \frac{\left(\frac{x+6}{4}\right)}{\left(\frac{2x-3}{x}\right)}$$

$$= \frac{x+6}{4} \cdot \frac{x}{2x-3}$$

$$= \frac{x(x+6)}{4(2x-3)}, \; x \neq 0$$

CHECKPOINT Now try Exercise 69.

In Example 8, the restriction $x \neq 1$ is added to the final expression to make its domain agree with the domain of the original expression.

Another way to simplify a complex fraction is to multiply each term in its numerator and denominator by the LCD of all fractions in its numerator and denominator. This method is applied to the fraction in Example 8 as follows.

Point out to your students that two different methods are shown here for simplifying the same complex fraction. Emphasize that both yield the same result.

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} \cdot \frac{x(x-1)}{x(x-1)}$$ LCD is $x(x-1)$.

$$= \frac{\left(\frac{2-3x}{\cancel{x}}\right) \cdot \cancel{x}(x-1)}{\left(\frac{x-2}{\cancel{x-1}}\right) \cdot x\cancel{(x-1)}}$$ Combine fractions.

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$ Simplify.

The next four examples illustrate some methods for simplifying rational expressions involving negative exponents and radicals. These types of expressions occur frequently in calculus.

To simplify an expression with negative exponents, one method is to begin by factoring out the common factor with the smaller exponent. Remember that when factoring, you subtract exponents. For instance, in $3x^{-5/2} + 2x^{-3/2}$ the smaller exponent is $-\frac{5}{2}$ and the common factor is $x^{-5/2}$.

$$\begin{aligned}3x^{-5/2} + 2x^{-3/2} &= x^{-5/2}[3(1) + 2x^{-3/2-(-5/2)}] \\ &= x^{-5/2}(3 + 2x^1) = \frac{3 + 2x}{x^{5/2}}\end{aligned}$$

Example 9 Simplifying an Expression with Negative Exponents

algebra of calculus

Simplify $x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2}$.

Solution

Begin by factoring out the common factor with the smaller exponent.

$$\begin{aligned}x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2} &= (1 - 2x)^{-3/2}[x + (1 - 2x)^{(-1/2)-(-3/2)}] \\ &= (1 - 2x)^{-3/2}[x + (1 - 2x)^1] \\ &= \frac{1 - x}{(1 - 2x)^{3/2}}\end{aligned}$$

CHECKPOINT Now try Exercise 75.

A second method for simplifying this type of expression involves multiplying the numerator and denominator by a term to eliminate the negative exponent.

Example 10 Simplifying an Expression with Negative Exponents

algebra of calculus

Simplify $\dfrac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2}$.

Solution

$$\begin{aligned}&\frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \\ &= \frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \cdot \frac{(4 - x^2)^{1/2}}{(4 - x^2)^{1/2}} \\ &= \frac{(4 - x^2)^1 + x^2(4 - x^2)^0}{(4 - x^2)^{3/2}} \\ &= \frac{4 - x^2 + x^2}{(4 - x^2)^{3/2}} = \frac{4}{(4 - x^2)^{3/2}}\end{aligned}$$

CHECKPOINT Now try Exercise 79.

Activities

1. What is the domain of $\sqrt{3 - x}$?
   Answer: The set of all real numbers less than or equal to 3.
2. The implied domain excludes what values of x from the domain of $\dfrac{x^2 + 6x + 9}{x^2 - 9}$?
   Answer: $x = 3, x = -3$
3. Simplify: $\dfrac{\left(\frac{1}{x} - \frac{1}{x + 1}\right)}{x + 1}$.
   Answer: $\dfrac{1}{x(x + 1)^2}$

Example 11 Rewriting a Difference Quotient

The following expression from calculus is an example of a *difference quotient*.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Rewrite this expression by rationalizing its numerator.

Solution

$$\begin{aligned}
\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\
&= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\
&= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \qquad h \neq 0
\end{aligned}$$

Notice that the original expression is undefined when $h = 0$. So, you must exclude $h = 0$ from the domain of the simplified expression so that the expressions are equivalent.

CHECKPOINT Now try Exercise 85.

Difference quotients, like that in Example 11, occur frequently in calculus. Often, they need to be rewritten in an equivalent form that can be evaluated when $h = 0$. Note that the equivalent form is not simpler than the original form, but it has the advantage that it is defined when $h = 0$.

Example 12 Rewriting a Difference Quotient

Rewrite the expression by rationalizing its numerator.

$$\frac{\sqrt{x-4}-\sqrt{x}}{4}$$

Solution

$$\begin{aligned}
\frac{\sqrt{x-4}-\sqrt{x}}{4} &= \frac{\sqrt{x-4}-\sqrt{x}}{4}\cdot\frac{\sqrt{x-4}+\sqrt{x}}{\sqrt{x-4}+\sqrt{x}}\\
&= \frac{\left(\sqrt{x-4}\right)^2-\left(\sqrt{x}\right)^2}{4\left(\sqrt{x-4}+\sqrt{x}\right)}\\
&= \frac{-4}{4\left(\sqrt{x-4}+\sqrt{x}\right)}\\
&= -\frac{1}{\sqrt{x-4}+\sqrt{x}}
\end{aligned}$$

CHECKPOINT Now try Exercise 86.

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P.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The set of real numbers for which an algebraic expression is defined is the _______ of the expression.
2. The quotient of two algebraic expressions is a fractional expression and the quotient of two polynomials is a _______ .
3. Fractional expressions with separate fractions in the numerator, denominator, or both are called _______ .
4. To simplify an expression with negative exponents, it is possible to begin by factoring out the common factor with the _______ exponent.
5. Two algebraic expressions that have the same domain and yield the same values for all numbers in their domains are called _______ .

In Exercises 1–16, find the domain of the expression.

1. $3x^2 - 4x + 7$
2. $2x^2 + 5x - 2$
3. $4x^3 + 3, \quad x \geq 0$
4. $6x^2 - 9, \quad x > 0$
5. $\dfrac{1}{3 - x}$
6. $\dfrac{x + 6}{3x + 2}$
7. $\dfrac{x^2 - 1}{x^2 - 2x + 1}$
8. $\dfrac{x^2 - 5x + 6}{x^2 - 4}$
9. $\dfrac{x^2 - 2x - 3}{x^2 - 6x + 9}$
10. $\dfrac{x^2 - x - 12}{x^2 - 8x + 16}$
11. $\sqrt{x + 7}$
12. $\sqrt{4 - x}$
13. $\sqrt{2x - 5}$
14. $\sqrt{4x + 5}$
15. $\dfrac{1}{\sqrt{x - 3}}$
16. $\dfrac{1}{\sqrt{x + 2}}$

In Exercises 17–22, find the missing factor in the numerator such that the two fractions are equivalent.

17. $\dfrac{5}{2x} = \dfrac{5(\quad)}{6x^2}$
18. $\dfrac{2}{3x^2} = \dfrac{2(\quad)}{3x^4}$
19. $\dfrac{3}{4} = \dfrac{3(\quad)}{4(x + 1)}$
20. $\dfrac{2}{5} = \dfrac{2(\quad)}{5(x - 3)}$
21. $\dfrac{x - 1}{4(x + 2)} = \dfrac{(x - 1)(\quad)}{4(x + 2)^2}$
22. $\dfrac{x + 3}{2(x - 1)} = \dfrac{(x + 3)(\quad)}{2(x - 1)^2}$

In Exercises 23–40, write the rational expression in simplest form.

23. $\dfrac{15x^2}{10x}$
24. $\dfrac{18y^2}{60y^5}$
25. $\dfrac{3xy}{xy + x}$
26. $\dfrac{2x^2y}{xy - y}$
27. $\dfrac{4y - 8y^2}{10y - 5}$
28. $\dfrac{9x^2 + 9x}{2x + 2}$
29. $\dfrac{x - 5}{10 - 2x}$
30. $\dfrac{12 - 4x}{x - 3}$
31. $\dfrac{y^2 - 16}{y + 4}$
32. $\dfrac{x^2 - 25}{5 - x}$
33. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - 4}$
34. $\dfrac{x^2 + 8x - 20}{x^2 + 11x + 10}$
35. $\dfrac{y^2 - 7y + 12}{y^2 + 3y - 18}$
36. $\dfrac{3 - x}{x^2 + 11x + 10}$
37. $\dfrac{2 - x + 2x^2 - x^3}{x - 2}$
38. $\dfrac{x^2 - 9}{x^3 + x^2 - 9x - 9}$
39. $\dfrac{z^3 - 8}{z^2 + 2z + 4}$
40. $\dfrac{y^3 - 2y^2 - 3y}{y^3 + 1}$

In Exercises 41 and 42, complete the table. What can you conclude?

41.

x	0	1	2	3	4	5	6
$\dfrac{x^2 - 2x - 3}{x - 3}$							
$x + 1$							

42.

x	0	1	2	3	4	5	6
$\dfrac{x - 3}{x^2 - x - 6}$							
$\dfrac{1}{x + 2}$							

43. ***Error Analysis*** Describe the error.

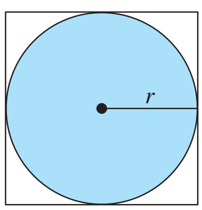
$$\frac{5x^3}{2x^3+4} = \frac{5x^3}{2x^3+4} = \frac{5}{2+4} = \frac{5}{6}$$

44. ***Error Analysis*** Describe the error.

$$\frac{x^3+25x}{x^2-2x-15} = \frac{x(x^2+25)}{(x-5)(x+3)}$$
$$= \frac{x(x-5)(x+5)}{(x-5)(x+3)} = \frac{x(x+5)}{x+3}$$

Geometry **In Exercises 45 and 46, find the ratio of the area of the shaded portion of the figure to the total area of the figure.**

45.



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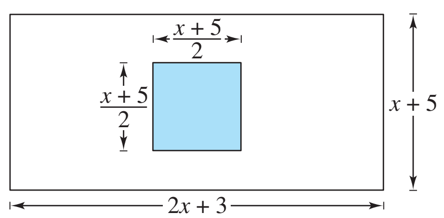


In Exercises 47–54, perform the multiplication or division and simplify.

47. $\dfrac{5}{x-1} \cdot \dfrac{x-1}{25(x-2)}$

48. $\dfrac{x+13}{x^3(3-x)} \cdot \dfrac{x(x-3)}{5}$

49. $\dfrac{r}{r-1} \div \dfrac{r^2}{r^2-1}$

50. $\dfrac{4y-16}{5y+15} \div \dfrac{4-y}{2y+6}$

51. $\dfrac{t^2-t-6}{t^2+6t+9} \cdot \dfrac{t+3}{t^2-4}$

52. $\dfrac{y^3-8}{2y^3} \cdot \dfrac{4y}{y^2-5y+6}$

53. $\dfrac{3(x+y)}{4} \div \dfrac{x+y}{2}$

54. $\dfrac{x+2}{5(x-3)} \div \dfrac{x-2}{5(x-3)}$

In Exercises 55–64, perform the addition or subtraction and simplify.

55. $\dfrac{5}{x-1} + \dfrac{x}{x-1}$

56. $\dfrac{2x-1}{x+3} - \dfrac{1-x}{x+3}$

57. $\dfrac{6}{2x+1} - \dfrac{x}{x+3}$

58. $\dfrac{3}{x-1} + \dfrac{5x}{3x+4}$

59. $\dfrac{3}{x-2} + \dfrac{5}{2-x}$

60. $\dfrac{2x}{x-5} - \dfrac{5}{5-x}$

61. $\dfrac{1}{x^2-x-2} - \dfrac{x}{x^2-5x+6}$

62. $\dfrac{2}{x^2-x-2} + \dfrac{10}{x^2+2x-8}$

63. $-\dfrac{1}{x} + \dfrac{2}{x^2+1} - \dfrac{1}{x^3+x}$

64. $\dfrac{2}{x+1} + \dfrac{2}{x-1} + \dfrac{1}{x^2-1}$

In Exercises 65–72, simplify the complex fraction.

65. $\dfrac{\left(\dfrac{x}{2}-1\right)}{(x-2)}$

66. $\dfrac{(x-4)}{\left(\dfrac{x}{4}-\dfrac{4}{x}\right)}$

67. $\dfrac{\left[\dfrac{x^2}{(x+1)^2}\right]}{\left[\dfrac{x}{(x+1)^3}\right]}$

68. $\dfrac{\left(\dfrac{x^2-1}{x}\right)}{\left[\dfrac{(x-1)^2}{x}\right]}$

69. $\dfrac{\left[\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}\right]}{h}$

70. $\dfrac{\left(\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}\right)}{h}$

71. $\dfrac{\left(\sqrt{x}-\dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

72. $\dfrac{\left(\dfrac{t^2}{\sqrt{t^2+1}}-\sqrt{t^2+1}\right)}{t^2}$

In Exercises 73–78, simplify the expression by removing the common factor with the smaller exponent.

73. $x^5 - 2x^{-2}$

74. $x^5 - 5x^{-3}$

75. $x^2(x^2+1)^{-5} - (x^2+1)^{-4}$

76. $2x(x-5)^{-3} - 4x^2(x-5)^{-4}$

77. $2x^2(x-1)^{1/2} - 5(x-1)^{-1/2}$

78. $4x^3(2x-1)^{3/2} - 2x(2x-1)^{-1/2}$

In Exercises 79–84, simplify the expression.

79. $\dfrac{2x^{3/2}-x^{-1/2}}{x^2}$

80. $\dfrac{x^2\left(\frac{3}{2}x^{-1/2}\right)-3x^{1/2}(x^2)}{x^4}$

81. $\dfrac{-x^2(x^2+1)^{-1/2}+2x(x^2+1)^{-3/2}}{x^3}$

82. $\dfrac{x^3(4x^{1/2})-3x^2\left(\frac{8}{3}x^{3/2}\right)}{x^6}$

83. $\dfrac{(x^2+5)\left(\frac{1}{2}\right)(4x+3)^{-1/2}(4)-(4x+3)^{1/2}(2x)}{(x^2+5)^2}$

84. $\dfrac{(2x+1)^{1/2}(3)(x-5)^2-(x-5)^3\left(\frac{1}{2}\right)(2x+1)^{-1/2}(2)}{2x+1}$

In Exercises 85–90, rationalize the numerator of the expression.

85. $\dfrac{\sqrt{x+2}-\sqrt{x}}{2}$

86. $\dfrac{\sqrt{z-3}-\sqrt{z}}{3}$

87. $\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

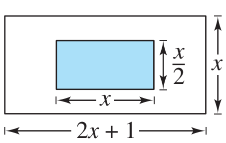
88. $\dfrac{\sqrt{x+5}-\sqrt{5}}{x}$

89. $\dfrac{\sqrt{x+9}-3}{x}$

90. $\dfrac{\sqrt{x+4}-2}{x}$

Probability **In Exercises 91 and 92, consider an experiment in which a marble is tossed into a box whose base is shown in the figure. The probability that the marble will come to rest in the shaded portion of the box is equal to the ratio of the shaded area to the total area of the figure. Find the probability.**

91.



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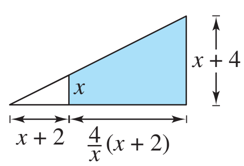

93. ***Rate*** A photocopier copies at a rate of 16 pages per minute.

(a) Find the time required to copy 1 page.

(b) Find the time required to copy x pages.

(c) Find the time required to copy 60 pages.

94. ***Monthly Payment*** The formula that approximates the annual interest rate r of a monthly installment loan is given by

$$r=\frac{\left[\dfrac{24(NM-P)}{N}\right]}{\left(P+\dfrac{NM}{12}\right)}$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

(a) Approximate the annual interest rate for a five-year car loan of $20,000 that has monthly payments of $400.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

95. ***Resistance*** The formula for the total resistance R_T (in ohms) of a parallel circuit is given by

$$R_T=\frac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_3}}$$

where R_1, R_2, and R_3 are the resistance values of the first, second, and third resistors, respectively.

(a) Simplify the total resistance formula.

(b) Find the total resistance in the parallel circuit when $R_1 = 6$ ohms, $R_2 = 4$ ohms, and $R_3 = 12$ ohms.

96. ***Refrigeration*** When food (at room temperature) is placed in a refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. Consider the model that gives the temperature of food that is at 75°F and is placed in a 40°F refrigerator as

$$T=10\left(\frac{4t^2+16t+75}{t^2+4t+10}\right)$$

where T is the temperature (in degrees Fahrenheit) and t is the time (in hours).

(a) Complete the table.

t	0	2	4	6	8	10
T						

t	12	14	16	18	20	22
T						

(b) What value of T does the mathematical model appear to be approaching?

97. ***Plants*** The table shows the numbers of endangered and threatened plant species in the United States for the years 2000 through 2005. (Source: U.S. Fish and Wildlife Service)

Year	Endangered, *E*	Threatened, *T*
2000	565	139
2001	592	144
2002	596	147
2003	599	147
2004	597	147
2005	599	147

Mathematical models for the data are

Endangered plants: $E = \dfrac{2342.52t^2 + 565}{3.91t^2 + 1}$

and

Threatened plants: $T = \dfrac{243.48t^2 + 139}{1.65t^2 + 1}$

where t represents the year, with $t = 0$ corresponding to 2000.

(a) Using the models, create a table to estimate the numbers of endangered plant species and the numbers of threatened plant species for the given years. Compare these estimates with the actual data.

(b) Determine a model for the ratio of the number of threatened plant species to the number of endangered plant species. Use the model to find this ratio for each of the given years.

98. ***Marriages and Divorces*** The table shows the rates (per 1000 of the total population) of marriages and divorces in the United States for the years 1990 through 2004. (Source: U.S. National Center for Health Statistics)

Year	Marriages, M	Divorces, D
1990	9.8	4.7
1991	9.4	4.7
1992	9.3	4.8
1993	9.0	4.6
1994	9.1	4.6
1995	8.9	4.4
1996	8.8	4.3
1997	8.9	4.3
1998	8.4	4.2
1999	8.6	4.1
2000	8.3	4.2
2001	8.2	4.0
2002	7.8	4.0
2003	7.5	3.8
2004	7.4	3.7

Mathematical models for the data are

Marriages: $M = \dfrac{8686.635t^2 - 191{,}897.18t - 9.8}{774.364t^2 - 20{,}427.65t - 1}$

and

Divorces: $D = -0.001t^2 - 0.06t + 4.8$

where t represents the year, with $t = 0$ corresponding to 1990.

(a) Using the models, create a table to estimate the number of marriages and the number of divorces for each of the given years. Compare these estimates with the actual data.

(b) Determine a model for the ratio of the number of marriages to the number of divorces. Use the model to find this ratio for each of the given years.

In Exercises 99–104, simplify the expression.

99. $\dfrac{(x + h)^2 - x^2}{h}, \quad h \neq 0$

100. $\dfrac{(x + h)^3 - x^3}{h}, \quad h \neq 0$

101. $\dfrac{\dfrac{1}{(x + h)^2} - \dfrac{1}{x^2}}{h}, \quad h \neq 0$

102. $\dfrac{\dfrac{1}{2(x + h)} - \dfrac{1}{2x}}{h}, \quad h \neq 0$

103. $\dfrac{\sqrt{2x + h} - \sqrt{2x}}{h}, \quad h \neq 0$

104. $\dfrac{\sqrt{x + h} - \sqrt{x}}{h}, \quad h \neq 0$

In Exercises 105 and 106, simplify the given expression.

105. $\dfrac{4}{n}\left(\dfrac{n(n + 1)(2n + 1)}{6}\right) + 2n\left(\dfrac{4}{n}\right)$

106. $9\left(\dfrac{3}{n}\right)\left(\dfrac{n(n + 1)(2n + 1)}{6}\right) - n\left(\dfrac{3}{n}\right)$

Synthesis

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. $\dfrac{x^{2n} - 1^{2n}}{x^n - 1^n} = x^n + 1^n$

108. $\dfrac{x^{2n} - n^2}{x^n - n} = x^n + n$

109. ***Think About It*** How do you determine whether a rational expression is in simplest form?

110. ***Think About It*** Is the following statement true for all nonzero real numbers a and b? Explain.

$$\frac{ax - b}{b - ax} = -1$$

111. ***Writing*** Write a paragraph explaining to a classmate why $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$.

112. ***Writing*** Write a paragraph explaining to a classmate why $\dfrac{1}{x + y} \neq \dfrac{1}{x} + \dfrac{1}{y}$.